THE IDEAL AND SUBIDEAL STRUCTURE OF LIE ALGEBRAS

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INTRODUCTION

In this thesis we study infinite-dimensional Lie algebras, drawing inspiration from group theory and ring theory.

Chapter one sets up notation.

Chapter two deals with prime ideals. In the first part of it, we define the concepts of a prime ideal and the radical of an ideal in Lie algebras along the same line as ideals in an associative rings,and investigate some of their properties. In the second we investigate the structure of Lie algebras with certain finiteness conditions on subideals, using the notion of prime ideals and prime algebras. In particular we prove that: If $\mathfrak{X}$ is one of Max-$\triangleleft^n$ ($n \geq 2$), Max-si, Min-$\triangleleft^n$ ($n \geq 2$), Min-si, then $L \in \mathfrak{X}$ if and only if:

(i) $\sigma(L)$ is a finite-dimensional soluble ideal of L

(ii) $L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $\mathfrak{X}$.

Chapter three deals with generalizations of the minimal condition on ideals, leading to a new class of "quasi-Artinian" algebras (We say that L is quasi-Artinian if for every descending chain of ideals $I_1 \supseteq I_2 \supseteq \dots$ of L there exists $m \in \mathbb{N}$ such that $[L^{(m)}, I_m] \subseteq I_n$ for all $n \geq m$) which possesses several of the main properties of Min-$\triangleleft$. In particular we prove that the class of quasi-Artinian algebras is Q-closed and a locally nilpotent quasi-Artinian Lie algebra is soluble.

Chapter four considers the join of subideals. First we prove that:

If L is a Lie algebra over a field of characteristic zero and if H_λ si L, $\lambda \in \Lambda$ such that $J = \langle H_\lambda | \lambda \in \Lambda \rangle$ and $\mathbb{B} = \{B | B < J,\ B \text{ si } L\}$, then J si L if and only if $\mathbb{B}$ has a maximal element. This result is a counterpart of a group-theoretic one (cf. Wielandt [35]). We also find another condition under which the join of subideals is a subideal by imposing conditions on the circle product $H \circ K = [H,K]^{\langle H,K \rangle}$ of subideals. In particular we show that:

If $\mathfrak{X}$ is an $\{I, N_0\}$-closed and locally coalescent class over any field, and if H and K are $\mathfrak{X}$-subideals of a Lie algebra L with $J = \langle H,K \rangle$ where $H \circ K | (H \circ K)^2$ is finitely generated, then J si L and $J \in \mathfrak{X}$.

Chapter five considers criteria for subideality and ascendancy generalizing some results of Kawamoto [17], Stitzinger [28]. Our main results are as follows:

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let $H < L$. Then H si L if and only if one of the following conditions holds:

(i) For each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x \rangle \circ \langle h_1 \rangle \circ \ldots \langle h_n \rangle \subseteq H$ for all $h_1, \ldots, h_n \in H$.

(ii) For each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x \rangle \circ_n \langle h \rangle \subseteq H$.

(iii) For each $h \in H$ there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x \rangle \circ_n \langle h \rangle \subseteq H$.

A generalization to infinite-dimensional Lie algebras leads to the following:

Let L be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H \prec L$. Then

(i) If for each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x\rangle \circ \langle h_1\rangle \circ \langle h_2\rangle \circ \ldots \circ \langle h_n\rangle \subseteq H$ for any $h_1,\ldots,h_n \in H$, then H asc L.

(ii) Suppose that H is finite-dimensional.

(a) If for each $h \in H$, there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x\rangle \circ_n \langle h\rangle \subseteq H$, then H si L.

(b) If for each $x \in L$ and $h \in H$, there exists an integer $n = n(x,h)$ such that $\langle x\rangle \circ_n \langle h\rangle \subseteq H$, then H asc L.

Finally if L is an ideally finite Lie algebra over a field of characteristic zero and if $H \prec L$, then H asc L if either of the following conditions holds:

(i) For each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x\rangle \circ_n \langle h\rangle \subseteq H$.

(ii) H asc $\langle H,x\rangle$ for each $x \in L$.

Some of these results are Lie-theoretic analogues of similar results for groups obtained by many authors, especially Wielandt [34] and Peng [22].

Chapter six considers subideals of the join of permutable Lie algebras.

First we consider subideals of the join of permutable finite-dimensional Lie algebras. Then we extend our results to certain classes of infinite dimension. Our main results are as follows:

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$ and $A \subseteq H$, $A \subseteq K$. Then A si L if and only if A si H and A si K.

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$. If H_i si $\langle H_i, H_j \rangle$ for all $i,j = 1,2,3$ and if $\langle H_1, H_2 \rangle$ is permutable with H_3. then H_i si L for all i.

Both results are counterparts of group-theoretic ones (see Wielandt [36,37]).

A generalization to infinite dimensions leads to the following:

Let L be a Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$ and $A \subseteq H$, $A \subseteq K$. Then

(a) If L is soluble-by-finite and A si H, A si K, then A si L.

(b) If L is ideally finite and A asc H, A asc K, then A asc L.

Finally, let L be a Lie algebra over any field and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$. Suppose that H_i si $\langle H_i, H_j \rangle = H_i + H_j$ for all $i,j = 1,2,3$ and $[H_1, H_2] \subseteq H_1$.

Then H_i si L for all i.

Two papers [1,2] based on this work have been accepted for publication, and two more [3,4] have been submitted.

All the results in this thesis are original except where explicitly stated otherwise.

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CHAPTER ONE : NOTATION AND TERMINOLOGY

All Lie algebras considered in this thesis will be of finite or infinite dimension, defined over an arbitrary field (unless otherwise stated). Our notation and terminology is based on Amayo and Stewart [5], but for the sake of convenience we will state the more important terms that we use here.

1.1. Preliminaries

Let L be a Lie algebra. By $H \subseteq L$, $H < L$, $H \triangleleft L$, we shall mean that H is a subset, subalgebra and ideal of L respectively. Angular brackets $< >$ denote the subalgebra generated by their contents. If X, Y are subspaces of L, then $[X,Y]$ is the subspace spanned by all products $[x,y]$, $x \in X$ and $y \in Y$. Let $x_1, x_2, \ldots, x_n \in L$. The left-normed products $[x_1,\ldots,x_n]$ are defined recursively by:
$[x_1,\ldots,x_{n+1}] = [[x_1,\ldots,x_n],x_{n+1}]$.

If $x_1 = x$, $x_2 = \ldots = x_{n+1} = y$, we write

$$[x,_n y] = [x_1,\ldots,x_{n+1}].$$

Similarly we define, for subsets $X_i \subseteq L$,

$$[X_1,\ldots,X_{n+1}] = [[X_1,\ldots,X_n],X_{n+1}]$$

and if $X = X_1$, $X_2 = X_3 = \ldots = X_{n+1} = Y$, we write

$$[X,_n Y] = [X_1,\ldots,X_{n+1}].$$

If $X, Y \subseteq L$, we say that X is Y-*invariant* if whenever $x \in X$ and $y \in Y$ then $[x,y] \in X$. Alternatively we say that Y *idealises* X. We let $\langle X^Y \rangle$ denote the smallest subalgebra of L which contains X and is Y-invariant and we call it the *ideal closure* of X under Y.

If $X \subseteq L$, then the *centralizer* of X in L is $C_L(X) = \{y \in L \mid [X,y] = 0\}$, and the *idealiser* of X in L is $I_L(X) = \{y \in L \mid [X,y] \subseteq X\}$.

We write $X \dot{+} Y$ to denote the *split extension* of an ideal X by a subalgebra Y (under a suitably specified Y-action on X).

1.2 Subideals

Let $H \leq L$. We say that H is a *subideal* of L if there is a finite series $H = H_0 \lhd H_1 \lhd \ldots \lhd H_n = L$. We write H si L.

To emphasize the role of the integer n we say that H is an n-*step subideal* of L, and write $H \lhd^n L$. We sometimes refer to n as the *subideal index* of H.

For any subalgebra $H \leq L$ we define the *ideal closure series* $(H_i)_{i \in \mathbb{N}}$ recursively by $H_0 = L$, $H_{i+1} = \langle H^{H_i} \rangle$. It follows that $H \lhd^n L$ if and only if $H_n = H$.

For any ordinal λ we say that H is a (λ-step) *ascendant subalgebra* of L if there is a series $(H_\alpha)_{\alpha \leq \lambda}$ of subalgebras of L, such that

(i) $H_0 = H$, $H_\lambda = L$

(ii) $H_\alpha \lhd H_{\alpha+1}$ if $\alpha < \lambda$

(iii) $H_\beta = \bigcup_{\alpha<\beta} H_\alpha$ if $\beta \leq \lambda$ is a limit ordinal. We write, H asc L, $H \triangleleft^\lambda L$.

1.3 Derivations

If $x \in L$ we define the *adjoint map* ad $x: L \to L$ by y ad $x = [y,x]$ $(y \in L)$.

From the Jacobi identity it follows that for any $y, z \in L, [y,z]\text{ad } x = [y \text{ ad} x,z] + [y,z\text{ad} x]$.

Any linear map $\delta: L \to L$ such that for all $y, z \in L$ $[y,z]\delta = [y\delta,z] + [y,z\delta]$ is called a *derivation* of L. Thus for any $x \in L$ the adjoint map ad x is a derivation, the *inner derivation* induced by x.

1.4 Central and Derived series

For an ordinal α we denote by L^α, $L^{(\alpha)}$, $\zeta_\alpha(L)$ the α-th terms of the (transfinite) *lower central*, *derived* and *upper central series* of L respectively, and define these inductively by $L^1 = L = L^{(0)}$, $L^{\alpha+1} = [L^\alpha,L]$, $L^{(\alpha+1)} = [L^{(\alpha)},L^{(\alpha)}]$ and at limit ordinals λ, $L^\lambda = \bigcap_{\alpha<\lambda} L^\alpha$ and $L^{(\lambda)} = \bigcap_{\alpha<\lambda} L^{(\alpha)}$;

we set $\zeta_0(L) = 0$, $\zeta_1(L) = C_L(L)$, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta_1(L/\zeta_\alpha(L))$ and for limit ordinals λ, $\zeta_\lambda = \bigcup_{\alpha<\lambda} \zeta_\alpha(L)$. We set

$\zeta_*(L) = \bigcup_\alpha \zeta_\alpha(L)$ and call it the *hypercentre*. The Lie algebra L is *hypercentral* if $L = \zeta_*(L)$.

L^{α}, $L^{(\alpha)}$, and $\zeta_{\alpha}(L)$ are all *characteristic ideals* of L in the sense that they are invariant under derivations of L. We write I ch L to mean that I is a characteristic ideal of L.

L is *nilpotent* (of *class* $\leq$ n) if $L^{n+1} = 0$, is *soluble* (of *derived length* $\leq$ n) if $L^{(n)} = 0$.

1.5 Classes and closure operations

A *class* $\mathfrak{X}$ of Lie algebras over a field F is a collection of Lie algebras over F such that

(i) $\mathfrak{X}$ contains the 0-dimensional subalgebra.

(ii) If $H \cong K \in \mathfrak{X}$, then $H \in \mathfrak{X}$.

Familiar classes of Lie algebras are:

$\mathfrak{F}$: finite-dimensional

$\mathfrak{A}$: abelian

$\mathfrak{N}$: nilpotent

$\mathfrak{N}_c$: nilpotent of class $\leq$ c ($c \in \mathbb{N}$).

Notation for other classes will follow.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes then $\mathfrak{X} \leq \mathfrak{Y}$ denotes inclusion. We denote by $\mathfrak{X}\mathfrak{Y}$ the class of all Lie algebras L with an $\mathfrak{X}$-ideal H such that $L/H \in \mathfrak{Y}$. We set $\mathfrak{X}_1 \dots \mathfrak{X}_{n+1} = (\mathfrak{X}_1 \dots \mathfrak{X}_n)\mathfrak{X}_{n+1}$ and write $\mathfrak{X}^{n+1}$ if all $\mathfrak{X}_i$'s equal $\mathfrak{X}$.

A *closure* operation A assigns to each class $\mathfrak{X}$ a class $A\mathfrak{X}$ in such a way that for all classes $\mathfrak{X}$ and $\mathfrak{Y}$ the following conditions hold:

(i) $A(0) = (0)$

(ii) $\mathfrak{X} \leq A\mathfrak{X}$

(iii) $A(A\mathfrak{X}) = A\mathfrak{X}$

(iv) $\mathfrak{X} \leq \mathfrak{Y} \Rightarrow A\mathfrak{X} \leq A\mathfrak{Y}$.

We say that $\mathfrak{X}$ is A-*closed* if $\mathfrak{X} = A\mathfrak{X}$.

We list some standard closure operations.

s : $s\mathfrak{X}$ consists of all subalgebras of $\mathfrak{X}$-algebras.

I : $I\mathfrak{X}$ consists of all subideals of $\mathfrak{X}$-algebras.

Q : $Q\mathfrak{X}$ consists of all quotients of $\mathfrak{X}$-algebras.

E : $E\mathfrak{X}$ consists of all algebras L having a finite series $0 = L_0 \triangleleft \dots \triangleleft L_n = L$ whose factors $L_{i+1}/L_i \in \mathfrak{X}$ for $0 \leq i \leq n-1$.

L : $L\mathfrak{X}$ consists of those algebras L such that every finite subset of L is contained in an $\mathfrak{X}$-subalgebra of L.

N_0: A class $\mathfrak{X}$ is N_0-closed if whenever $H, K \triangleleft L$ and $H, K \in \mathfrak{X}$ then $H + K \in \mathfrak{X}$.

R : $R\mathfrak{X}$ consists of those algebras L having a family $(I_\alpha)_{\alpha \in A}$ of ideals such that $L/I \in \mathfrak{X}$ for all $\alpha \in A$ and $\bigcap_{\alpha \in A} I_\alpha = 0$.

The operations E, L, R are read as «poly», «locally», «residually» respectively. In particular we note the classes

$E\mathfrak{A}$: soluble

$L\mathfrak{N}$: locally nilpotent

$L\mathfrak{F}$: locally finite (-dimensiona)

$R\mathfrak{N}$: residually nilpotent

$R\mathfrak{F}$: residually finite (-dimensional).

If A and B are operations we define AB by:

$$AB\mathfrak{X} = A(B\mathfrak{X}).$$

In general AB need not be a closure operation. However, let us define an ordering on operations by

$$A \leq B \Longleftrightarrow A\mathfrak{X} \leq B\mathfrak{X} \text{ for all classes } \mathfrak{X}.$$

If $BA \leq AB$ then it is easy to see that AB is a closure operation, and is in fact equal to $\{A,B\}$.

1.6 Chain conditions

Let V be a vector space, and $\mathcal{S}$ a collection of subsets of V. We say that V has (or satisfies) Max-$\mathcal{S}$ if $\mathcal{S}$ satisfies the maximal condition: every ascending chain $S_0 \subseteq S_1 \subseteq \cdots$ of elements $S_i \in \mathcal{S}$ terminates finitely; so that $S_r = S_{r+1} = \cdots$ for some $r \in \mathbb{N}$.

Similarly V has Min-$\mathcal{S}$ if $\mathcal{S}$ satisfies the minimal condition: every descending chain $S_0 \supseteq S_1 \supseteq \cdots$ terminates.

If V is a Lie algebra L and $\mathcal{S}$ is respectively the set of ideals, subideals, n-step subideals of L we write in place of Max-$\mathcal{S}$

$$\text{Max-}\triangleleft,\ \text{Max-si},\ \text{Max-}\triangleleft^n$$

and for Min-$\mathcal{S}$ we write

$$\text{Min-}\triangleleft,\ \text{Min-si},\ \text{Min-}\triangleleft^n.$$

We use the same notation for the classes of Lie algebras satisfying the corresponding conditions.

CHAPTER TWO : PRIME IDEALS IN LIE ALGEBRAS

The notion of prime ideals plays an important role in the theory of associative algebras. It is of some interest to know how the corresponding notion behaves in Lie algebras. In the first section of this chapter, which is based on certain results of Kawamoto [18], we define the concepts of prime ideal and radical of an ideal in Lie algebras along the same lines as ideals in an associative ring, and investigate some of their properties. The main result of this section states that: If $L \in \text{Max-}\triangleleft$ and $I \triangleleft L$, then there exist a finite number of minimal prime ideals belonging to I. (Theorem 2.1.13). In Section two of this chapter, which is based on [1], we investigate the structure of Lie algebras with certain finiteness conditions on subideals. The main result states that: If $\mathfrak{X}$ is one of $\text{Max-}\triangleleft^n$, $n \geq 2$, Max-si, $\text{Min-}\triangleleft^n$, $n \geq 2$, Min-si, then $L \in \mathfrak{X}$ if and only if:

(i) $\sigma(L) \in \mathfrak{F} \cap \mathfrak{E}\mathfrak{A}$

(ii) $L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $\mathfrak{X}$. (Theorem 2.2.3)

2.1 Prime ideals and radicals

This section is based on certain results of Kawamoto [18], Behrens [7] and McCoy [21]. The proofs are closely related to Behrens [7] and McCoy [21]. We start with the following:

Definition 2.1.1

An ideal P of L is a *prime* ideal of L if whenever $[\langle a^L\rangle, \langle b^L\rangle] \subseteq P$ at least one of a and b belongs to P.

From this definition we have:

Proposition 2.1.2

Let P be an ideal of L. Then P is prime if and only if $[A,B] \subseteq P$ with A,B ideals of L implies $A \subseteq P$ or $B \subseteq P$.

Proof

Let A, B be ideals of L such that $[A,B] \subseteq P$ and $A \not\subseteq P$. Suppose $a \in A$ with $a \notin P$, and that b is an arbitrary element of B. Since $[\langle a^L\rangle, \langle b^L\rangle] \subseteq [A,B] \subseteq P$ with $a \notin P$, it follows that $b \in P$. Hence $B \subseteq P$.

The converse is clear. □

To give another characterization of a prime ideal P, we shall consider the set-theoretic complement C(P) of P in L. This is an m-system in the following sense.

Definition 2.1.3 A subset M of L is an m-*system*, if for any $x,y \in M$ there exist in $\langle x^L\rangle$ and $\langle y^L\rangle$ two elements x_1 and y_1 respectively such that $[x_1,y_1] \in M$. The empty set $\emptyset$ is to be considered an m-system.

This concept plays the same role as the analogous one defined by Behrens [7] and so we can translate some of his results into ours. First we have from Definition 2.1.1, that an ideal P in L is a prime ideal of L if and only if its complement C(P) is an m-system.

Definition 2.1.4

A prime ideal P is called a *minimal prime ideal* belonging to an ideal I, if $P \supseteq I$ and there is no prime ideal P_1 of L such that $I \subseteq P_1 \subsetneq P$.

The *radical* rad(I) of an ideal I in L is the intersection of all minimal prime ideals belonging to I. We write rad(L) for rad(0).

Next the following.

Theorem 2.1.5

Let $I \lhd L$. Then

(i) rad(I) consists of those elements x of L with property that every m-system which contains x contains an element of I.

(ii) rad(I) is the intersection of all prime ideals containing I.

The proof follows after a series of lemmas.

Lemma 2.1.6

Let $I \lhd L$ and let M be an m-system such that $I \cap M = \emptyset$. Then M is contained in an m-system M* which is maximal in the class of m-systems which do not intersect I.

Proof

Immediate consequence of Zorn's Lemma. $\square$

Lemma 2.1.7

Let M be an m-system in L and let $I \lhd L$ be such that $I \cap M = \emptyset$. Then I is contained in an ideal P of L which is maximal in the class of ideals which do not intersect M. The ideal P is necessarily a prime ideal of L.

Proof

The existence of P follows from Zorn's lemma. We now show that P is a prime ideal of L. If $M = \emptyset$, then $P = L$ and P is a prime ideal of L. Suppose that $M \neq \emptyset$ and A, B are ideals of L such that $A \not\subseteq P$, $B \not\subseteq P$. Then the maximality of P implies that A + P contains an elements x of M and B + P contains an element y of M. Since M is an m-system there exist $x_1 \in \langle x^L \rangle$ and $y_1 \in \langle y^L \rangle$ such that $[x_1,y_1] \in M$. Moreover $[x_1,y_1] \in [A + P, B + P]$. Now if $[A,B] \subseteq P$, we would have $[A + P, B + P] \subseteq P$ and it follows that $[x_1,y_1] \in P$. But this is impossible since $[x_1,y_1] \in M$ and $M \cap P = \emptyset$. Hence $[A,B] \not\subseteq P$ and P is therefore a prime ideal of L. $\square$

Lemma 2.1.8

A set P of elements of L is a minimal prime ideal belonging to an ideal I in L if and only if its complement C(P) is maximal in the class of m-systems which do not intersect I.

Proof

Let P be a set of elements of L with the property that $M = C(P)$ is maximal among the set of m-systems which do not intersect I. By Lemma 2.1.7, there is a prime ideal P^* which contains I and $P^* \cap M = \emptyset$. Hence $C(P^*)$ is an m-system which does not intersect I and contains M. From the maximal property of M, it follows that $C(P^*) = M$, and hence $P = P^*$. Also this maximal property shows that there is no prime ideal P_1 such that $I \subseteq P_1 \subsetneq P$, as otherwise $C(P_1)$ would be an m-system which does not intersect I and contains M as proper subset. Hence P is a minimal prime ideal belonging to I.

Conversely, if P is a minimal prime ideal belonging to I, then $M = C(P)$ is an m-system which does not intersect I, and Lemma 2.1.6 shows the existence of a maximal m-system M^* which contains M and does not intersect I. By the case just proved, $C(M^*) = P^*$ is a minimal prime ideal belonging to I. But since $M^* \supseteq M$, it follows that $P^* \subseteq P$, and the minimal property of P then shows that $P = P^*$. Hence $M = M^*$, and M is a maximal m-system which does not interesect I. □

Proof of Theorem 2.1.5

(i) Suppose that there exists an m-system M such that $x \in M$, but $M \cap I = \emptyset$. By Lemma 2.1.6, M is contained in an m-system M^* which is maximal in the class of m-systems which do not intersect I. By Lemma 2.1.8, $C(M^*)$ is a minimal prime ideal belonging to I, and clearly C(M*) does not contain x. Hence $x \notin \mathrm{rad}(I)$.

Conversely, if P is any minimal prime ideal belonging to I, then C(P) is an m-system which does not intersect I, and hence $x \notin C(P)$ by our assumption, that is, $x \in P$. Thus $x \in \mathrm{rad}(I)$.

(ii) It is enough to show that every prime ideal contains I contains a minimal prime ideal belonging to I. So suppose that P be a prime ideal of L containing I. Then C(P) is an m-system which does not intersect I. By Lemma 2.1.6, C(P) is contained in an m-system M* which is maximal in the class of m-systems which do not intersect I. Lemma 2.1.8 shows that C(M*) is a minimal prime ideal belonging to I. Since $C(P) \subseteq M^*$, it follows that $I \subseteq C(M^*) \subseteq P$. □

Next we shall collect certain results on ideals under the assumption that L satisfies the maximal condition for ideals, but first we recall the following.

<u>Definition 2.1</u>.9

Let L be a Lie algebras. Then $\sigma(L)$ is defined to be the sum of the soluble ideals of L (see Amayo and Stewart [5, p. 180]. We say L is *semi-simple* if $\sigma(L) = 0$.

<u>Proposition 2.1.</u>10

Let $L \in \mathrm{Max}\text{-}\triangleleft$ and let $I \triangleleft L$. Then there exists $n \in \mathbb{N}$ such that $(\mathrm{rad}(I))^{(n)} \subseteq I$.

To prove this we need the following lemmas. The first one is Kawamoto [18, Theorem 7]. We give the proof for completeness.

Lemma 2.1.11

$\sigma(L) \subseteq \mathrm{rad}(L)$. If $L \in \text{Max-}\triangleleft$, then $\sigma(L) = \mathrm{rad}(L)$.

Proof

Let I be a soluble ideal of L. Then there exists $n \in \mathbb{N}$ such that $I^{(n)} = 0$. For any prime ideal P of L we have $I \subseteq P$ since $I^{(n)} = 0 \subseteq P$. Therefore $\sigma(L) \subseteq \mathrm{rad}(L)$.

If $L \in \text{Max-}\triangleleft$, then $\sigma(L)$ is the unique maximal soluble ideal of L. Assume that $\mathrm{rad}(L) = R$ is not soluble. Let $\mathbf{C}$ be the collection of ideals I such that $R^{(n)} \not\subseteq I$ for all $n \geq 0$. $\mathbf{C} \neq \emptyset$ because $0 \in \mathbf{C}$. Hence $\mathbf{C}$ has a maximal element P. We claim that P is prime. If there are ideals A, B of L such that $A \not\subseteq P$, $B \not\subseteq P$ and $[A,B] \subseteq P$, then $A + P, B + P \notin \mathbf{C}$ by definition of P. Hence $R^{(n)} \subseteq A + P$, $R^{(m)} \subseteq B + P$ for some $m, n \in \mathbb{N}$. Let $s = \max \{n,m\}$. Then $R^{(s+1)} \subseteq [A+P,B+P] \subseteq P$. But this contradicts $P \in \mathbf{C}$. Therefore R is soluble and $\mathrm{rad}(L) = \sigma(L)$, which completes the proof. □

Lemma 2.1.12

Let $I \triangleleft L$. Then $\mathrm{rad}(L/I) = \mathrm{rad}(I)/I$.

Proof It follows from Theorem 2.1.5, that $\mathrm{rad}(L/I) = \cap\{P/I : P$ is a prime ideal of L containing $I\}$ and hence

$\text{rad}(L/I) = (\cap \{P: P \text{ is a prime ideal of } L \text{ containing } I\})/I$.
Thus $\text{rad}(L/I) = \text{rad}(I)/I$. □

Proof of Proposition 2.1.10

By Lemma 2.1.12, we have $\text{rad}(L/I) = \text{rad}(I)/I$. But by Lemma 2.1.11, $\text{rad}(L/I) = \sigma(L/I)$ which is soluble, hence $\text{rad}(I)/I$ is soluble and so there exists $n \in \mathbb{N}$ such that $(\text{rad}(I))^{(n)} \subseteq I$. □

Finally we prove the following.

Theorem 2.1.13

If $L \in \text{Max-}\triangleleft$ and $I \triangleleft L$, then there exists a finite number of minimal prime ideals belonging to I. Thus rad(I) is an intersection of a finite number of minimal prime ideals belonging to I.

Proof

If I is a prime ideal of L, then the assertion is trivial. We may suppose therefore that I is not a prime ideal of L. Then we can find ideals A, B of L such that $A \not\subseteq I$ and $B \not\subseteq I$ but $[A,B] \subseteq I$. Let us suppose that I has an infinite number of minimal prime ideals P_i belonging to it. Then since $[A+I, B+I] \subseteq I$ at least one of $A + I$ and $B + I$ must be contained in an infinite number of P_i. Without loss of generality we may assume that the one which is contained in an infinite number of P_i is $A + I$. It is easily seen that those P_i which contain $A + I$ are minimal prime ideals belonging to $A + I$ and moreover $A + I \supsetneq I$. Continuing an exactly similar argument, we obtain a strictly increasing sequence $I \subsetneq A + I \subsetneq \dots$ of ideals of L, which is impossible from our assumption. □

2.2 On Lie algebras with finiteness conditions

The object of this section is to investigate the structure of Lie algebras with certain finiteness conditions on subideals, using the notion of prime ideals and prime algebras.

We start with the following definition which is the Lie analogue of prime ring.

Definition 2.2.1

We say that a Lie algebra L is *prime* if whenever I and J are ideals of L and $[I,J] = 0$, then either $I = 0$ or $J = 0$.

It follows that P is a prime ideal of L if and only if L/P is a prime algebra.

Definition 2.2.2

A Lie algebra L is said to be a *subdirect sum* of a family of Lie algebras $\{L_\alpha\}_{\alpha\in A}$ if there is an injective homomorphism $f: L \longrightarrow \sum_{\alpha\in A} \oplus L_\alpha$ such that for each $\beta \in A$ $\varepsilon_\beta \circ f: L \longrightarrow L_\beta$ is a surjective homomorphism where ε_β is the projection of $\sum_{\alpha\in A} \oplus L_\alpha$ onto L_β.

The main result in this section is:

Theorem 2.2.3

Let L be a Lie algebra and let $\mathfrak{X}$ be one of Max-$\lhd^n$ $(n \geq 2)$, Max-si, Min-$\lhd^n$ $(n \geq 2)$, Min-si. Then $L \in \mathfrak{X}$ if and only if:

(i) $\sigma(L)$ is a finite-dimensional soluble ideal of L.

(ii) $L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $\mathfrak{X}$.

The proof follows from a series of lemmas.

Lemma 2.2.4 Let $I \lhd L$ and H be a subideal (resp. n-step subideal) of L. Then $H \cap I$ is a subideal (resp. n-step subideal) of L.

Proof

Let $H = H_n \lhd H_{n-1} \lhd \dots \lhd H_1 \lhd H_0 = L$ be the ideal closure series of H in L. Then $H \cap I = H_n \cap I \lhd H_{n-1} \cap I \lhd \dots \lhd H_1 \cap I \lhd L$ and so $H \cap I \lhd^n L$. $\square$

Lemma 2.2.5

Let L be a Lie algebra and let $\mathcal{S}$ be respectively the set of ideals, subideals, n-step subideals of L. Suppose $I_i \lhd L$, $(i = 1,2,\dots,m)$ and $\bigcap_{i=1}^{m} I_i = 0$. Let $\mathcal{S}_i = \{\frac{H+I_i}{I_i} | H \in \mathcal{S}\}$. If $L/I_i \in$ Max-$\mathcal{S}_i$ (resp. Min-$\mathcal{S}_i$) for all i, then $L \in$ Max-$\mathcal{S}$ (resp. Min-$\mathcal{S}$).

Proof

By induction on m we need consider only the case $m = 2$, then $I_1 \cap I_2 = 0$. Let $H_1 \subseteq H_2 \subseteq \dots$ be an ascending chain of elements $H_i \in \mathfrak{S}$. Then $(H_1+I_1)/I_1 \subseteq (H_2+I_1)/I_1 \subseteq \dots$ is an ascending chain of elements of $\mathfrak{S}_1$. Therefore there exists $r \in \mathbb{N}$ such that $H_r + I_1 = H_{r+1} + I_1 = \dots \quad \dots$ (1)
Now $H_1 \cap I_1 \subseteq H_2 \cap I_1 \subseteq \dots$ is an ascending chain of elements of $\mathfrak{S}$ by Lemma 2.2.4. Therefore $\frac{(H_1 \cap I_1) + I_2}{I_2} \subseteq \frac{(H_2 \cap I_1) + I_2}{I_2} \subseteq \dots$ is an ascending chain of elements of $\mathfrak{S}_2$ and so there exists $r \in \mathbb{N}$ such that $(H_r \cap I_1) + I_2 = (H_{r+1} \cap I_1) + I_2 = \dots$. Hence $H_{r+1} \cap I_1 = (H_r \cap I_1) + H_{r+1} \cap I_1 \cap I_2$ by the modular law, but $I_1 \cap I_2 = 0$, therefore $H_{r+1} \cap I_1 = H_r \cap I_1 \dots$ (2)
Now from (1) and (2) we have $H_r = H_{r+1} = \dots$ and so $L \in$ Max-$\mathfrak{S}$. That $L \in$ Min-$\mathfrak{S}$ can be proved by a similar method. □

Lemma 2.2.6

Let L be a Lie algebra and $\{L_\alpha\}_{\alpha \in A}$ be a family of Lie algebras. Then L is a subdirect sum of $\{L_\alpha\}_{\alpha \in A}$ if and only if for each $\beta \in A$, there is a surjective homomorphism $g_\beta : L \longrightarrow L_\beta$ such that $\bigcap_{\beta \in A} \operatorname{Ker}.g_\beta = 0$.

Proof:

This can be proved in the same way as in Gray [12, p.88]. □

An immediate consequence of Lemma 2.2.6 we have the following:

Corollary 2.2.7

Let L be a Lie algebra and let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals of L. If $\bigcap_{\alpha \in A} I_\alpha = 0$, then L is a subdirect sum of the family of Lie algebras $\{L/I_\alpha\}_{\alpha \in A}$.

Lemma 2.2.8

If $L \in \text{Max-}\triangleleft^2$ (resp. $\text{Min-}\triangleleft^2$), then $\sigma(L)$ is a finite-dimensional soluble ideal of L.

Proof

This follows from Amayo and Stewart [5, Corollary 9.1.3(d), p. 183 and Lemma 9.2.1, p. 190]. □

Lemma 2.2.9

Let L be a Lie algebra.

(i) L is semi-simple with $\text{Max-}\triangleleft^n$, $n \geq 1$ (resp. Max-si) if and only if L is a subdirect sum of a finite number of prime algebras satisfying $\text{Max-}\triangleleft^n$, $n \geq 1$ (resp. Max-si).

(ii) L is semi-simple with $\text{Min-}\triangleleft^n$, $n \geq 1$ (resp. Min-si) if and only if L is a subdirect sum of a finite number of prime algebras satisfying $\text{Min-}\triangleleft^n$ $(n \geq 1)$ (resp. Min-si).

Proof

(i) Let L be semi-simple with $\text{Max-}\triangleleft^n$ (resp. Max-si). Then $\sigma(L) = 0$. By Theorem 2.1.13, $\text{rad}(L) = \bigcap_{i=1}^{m} P_i$, where the P_i are prime ideals of L. But by Lemma 2.1.11, $\text{rad}(L) = \sigma(L)$, hence $\text{rad}(L) = 0$. Since P_i is a prime ideal of L, it follows that L/P_i is a prime algebra and $L/P_i \in$ Max-$\triangleleft^n$ (resp. Max-si). Now by Corollary 2.2.7, L is a subdirect sum of a finite number of prime algebras satisfying Max-$\triangleleft^n$, (resp. Max-si). To prove the converse suppose that L is a subdirect sum of a finite number of prime algebras $\{L_\alpha\}_{\alpha\in A}$, $A = \{1,2,\ldots,m\}$ satisfying Max-$\triangleleft^n$ (resp. Max-si). Let $g_\beta : L \longrightarrow L_\beta$ be the surjective homomorphism of Lemma 2.2.6. Then for each β, $L/\mathrm{Ker}\, g_\beta \simeq L_\beta$ and L_β is prime. Hence $\mathrm{Ker}\, g_\beta$ is a prime ideal of L. Thus $\mathrm{rad}(L) \subseteq \mathrm{Ker}\, g_\beta$ for each β, and so $\mathrm{rad}(L) \subseteq \bigcap_{\beta\in A} \mathrm{Ker}\, g_\beta = 0$. But by Lemma 2.1.11, $\sigma(L) \subseteq \mathrm{rad}(L)$, hence $\sigma(L) = 0$ and L is semi-simple. Now that $L \in$ Max-$\triangleleft^n$ (resp. Max-si) follows from Lemma 2.2.5.

(ii) Let L be semi-simple with Min-$\triangleleft^n$ (resp. Min-si). Then L has only a finite number of minimal ideals $M_1,\ldots,M_r$. Let P_i, $1 \le i \le r$, be an ideal of L which is maximal with respect to not containing M_i. We claim that P_i is a prime ideal of L. Suppose not. Then there exists ideals I, J of L such that $I \not\subseteq P_i$, $J \not\subseteq P_i$ and $[I,J] \subseteq P_i$. Now $I + P_i \supsetneq P_i$ and $J + P_i \supsetneq P_i$, so by the choice of P_i, $I + P_i \supset M_i$ and $J + P_i \supset M_i$. Therefore $M_i^2 \subseteq [I + P_i, J + P_i] \subseteq P_i$. But $M_i^2 \neq 0$ for L is semi-simple, hence $M_i^2 = M_i \subseteq P_i$ which is a contradiction. Therefore P_i is a prime ideal of L and L/P_i is a prime algebra. If $\bigcap_{i=1}^{r} P_i \neq 0$, then this intersection contains one of the minimal ideals M_j for some j. But $M_j \not\subseteq P_j$, so $M_j \not\subseteq \bigcap_{i=1}^{r} P_i$. Hence $\bigcap_{i=1}^{r} P_i = 0$ and by Corollary 2.2.7, it follows that L is a subdirect sum of a finite number of prime algebras satisfying Min-$\triangleleft^n$ (resp. Min-si).

Conversely that L is semi-simple can be proved as in (i), and that $L \in$ Min-$\triangleleft^n$ (resp. Min-si) follows from Lemma 2.2.5 □

Proof of Theorem 2.3

The result follows from Lemma 2.2.8 and Lemma 2.2.9. □

Theorem 2.2.3 reduces several problems about algebras with chain conditions to the case of prime algebras. We intend to make use of this reduction in future work.

so $M_j \not\subseteq \bigcap_{i=1}^{r} P_i$. Hence $\bigcap_{i=1}^{r} P_i = 0$ and by Corollary 2.2.7, it follows that L is a subdirect sum of a finite number of prime algebras satisfying Min-$\triangleleft^n$ (resp. Min-si).

Conversely that L is semi-simple can be proved as in (i), and that $L \in$ Min-$\triangleleft^n$ (resp. Min-si) follows from Lemma 2.2.5 □

<u>Proof of Theorem 2.3</u>

The result follows from Lemma 2.2.8 and Lemma 2.2.9. □

Theorem 2.2.3 reduces several problems about algebras with chain conditions to the case of prime algebras. We intend to make use of this reduction in future work.

CHAPTER THREE : QUASI-ARTINIAN LIE ALGEBRAS

Stewart ([25], pp. 90-92) has shown that the class of Artinian Lie algebras (Lie algebras with Min-$\triangleleft$) is $\{Q,E\}$ closed. Also he shows that a locally nilpotent Artinian Lie algebra is soluble. In this chapter, which is based on [1], we introduce the notion of Quasi-Artinian Lie algebras which generalizes the Artinian Lie algebras in such a way that its main properties are preserved.

Definition 3.1

We say that a Lie algebra L is *quasi-Artinian* if for every descending chain $I_1 \supseteq I_2 \supseteq \ldots$ of ideals of L there exist $r,s \in \mathbb{N}$ such that $[L^{(r)},I_s] \subseteq I_n$ for all n, or equivalently there exists $m \in \mathbb{N}$ such that $[L^{(m)},I_m] \subseteq I_n$ for all n.

It is clear that every soluble Lie algebra is quasi-Artinian, but it is easy to construct a soluble Lie algebra which is not Artinian, so quasi-Artinian algebras need not be Artinian. Further we have the following:

Proposition 3.2

If L is a hypercentral Lie algebra and is quasi-Artinian, then L is soluble.

To prove this we need the following well-known result.

Lemma 3.3

If L is hypercentral then $L^{(\alpha)} = 0$ for some ordinal α.

Proof

See Amayo and Stewart [5, Lemma 8.1.1, p. 163]. □

Proof of Proposition 3.2

By Lemma 3.3, $L^{(\alpha)} = 0$ for some ordinal α. But $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$ is a descending chain of ideals of L and L is quasi-Artinian, so there exists $m \in \mathbb{N}$ such that $[L^{(m)}, L^{(m)}] \subseteq L^{(n)}$ for all n. Hence $L^{(m+1)} = L^{(n)}$ for all n. Therefore $L^{(m+1)} = L^{(m+2)} = \ldots$ and $L^{(\alpha)} = L^{(m+1)} = 0$. Thus L is soluble. □

Theorem 3.4

The following are equivalent:

(i) L is quasi-Artinian.

(ii) There exists $m \in \mathbb{N}$ such that for every descending chain $I_1 \supseteq I_2 \supseteq \ldots$ of ideals of L, the descending chain of ideals $[L^{(m)}, I_1] \supseteq [L^{(m)}, I_2] \supseteq \ldots$ terminates.

(iii) For every non-empty collection $\mathfrak{C}$ of ideals of L, there exists an element $I \in \mathfrak{C}$ and $m \in \mathbb{N}$ such that $[L^{(m)}, I] \subseteq J$ for every $J \in \mathfrak{C}$ with $J \subseteq I$.

Proof

(i) $\Rightarrow$ (ii). Let L be quasi-Artinian. Now $L \supseteq L^{(1)} \supseteq \ldots$

is a descending chain of ideals of L, so there exists $m \in \mathbb{N}$ such that $[L^{(m)},L^{(m)}] \subseteq L^{(n)}$ for all $n \geq m$. Therefore $L^{(m+1)} = L^{(m+2)} = \dots$. Also $I_1 \supseteq I_2 \supseteq \dots$ is a descending chain of ideals of L, so there exists $r \in \mathbb{N}$ such that $[L^{(2m)},I_r] \subseteq I_{r+s}$ for all $s \in \mathbb{N}$. Therefore for all $s \in \mathbb{N}$, $[L^{(2m)},I_r] \subseteq [L^{(2m)}, [L^{(2m)},I_r]] \subseteq [L^{(2m)},I_{r+s}] \subseteq [L^{(2m)},I_r]$. Hence $[L^{(2m)},I_r] = [L^{(2m)},I_{r+s}]$. Since the choice of m is independent of the sequence $\{I_n\}$, the result follows.

(ii) $\Rightarrow$ (iii). Suppose that (iii) does not hold for some $\mathfrak{C}$. Then we can successively find $I_i \in \mathfrak{C}$, $(i = 1,2,\dots)$ such that $I_i \supset I_{i+1}$, but $[L^{(i)},I_i] \not\subseteq I_{i+1}$ which implies that (ii) does not hold. Hence (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) is clear. □

<u>Theorem 3.5</u>

(i) Let L be a quasi-Artinian Lie algebra and let $I \triangleleft L$. Then L/I is quasi-Artinian.

In other words the class of quasi-Artinian algebras is Q-closed.

(ii) Let $I \triangleleft L$. Then L is quasi-Artinian if one of the following holds:

(a) I is quasi-Artinian and L/I is soluble.

(b) L/I is quasi-Artinian and if $I \supseteq I_1 \supseteq I_2 \supseteq \dots$, $I_i \triangleleft L$ then there exists $m \in \mathbb{N}$ such that $[L^{(m)},I_m] \subseteq I_n$ for all $n \in \mathbb{N}$.

(c) L/I is quasi-Artinian and I is Artinian.

<u>Proof</u>

(i) Let $\pi : L \longrightarrow L/I$ be the natural homomorphism and let $\bar{I}_1 \supseteq \bar{I}_2 \supseteq \ldots$ be a descending chain of ideals of $\bar{L} = L/I$. Then $\pi^{-1}(\bar{I}_1) \supseteq \pi^{-1}(\bar{I}_2) \supseteq \ldots$ is a descending chain of ideals of L. But L is quasi-Artinian, so there exists $m \in \mathbb{N}$ such that $[L^{(m)}, \pi^{-1}(\bar{I}_m)] \subseteq \pi^{-1}(\bar{I}_n)$ for all $n \geq m$. Therefore $[(\pi(L))^{(m)}, \bar{I}_m] = \pi[L^{(m)}, \pi^{-1}(\bar{I}_m)] \subseteq \bar{I}_n$. Thus $[\bar{L}^{(m)}, \bar{I}_m] \subseteq \bar{I}_n$ and I is quasi-Artinian.

(ii) (a), (b) let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of ideals of L. Then $I_1 \cap I \supseteq I_2 \cap I \supseteq \ldots$ is a descending chain of ideals of I and $(I_1 + I)/I \supseteq (I_2 + I)/I \supseteq \ldots$ is a descending chain of ideals of L/I. By assumption (a) or (b), there exists $m \in \mathbb{N}$ such that $[L^{(m)}, I_m \cap I] \subseteq I_n \cap I$ and $[L^{(m)}, I_m] + I \subseteq [L^{(m)}, I_n] + I$ for all $n \geq m$. Therefore $[L^{(m)}, I_m] \subseteq I_n + I$, but $[L^{(m)}, I_m] \subseteq I_m$. Hence $[L^{(m)}, I_m] \subseteq (I_n + I) \cap I_m = I_n + (I_m \cap I)$ and so $[L^{(m)}, [L^{(m)}, I_m]] \subseteq [L^{(m)}, I_n] + [L^{(m)}, (I_m \cap I)]$. Therefore $[L^{(m+1)}, I_m] \subseteq I_n + (I_n \cap I) = I_n$ and L is quasi-Artinian.

(c) Let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of ideals of L. Then $I_1 \cap I \supseteq I_2 \cap I \supseteq \ldots$ is a descending chain of ideals of I and $(I_1 + I)/I \supseteq (I_2 + I)/I \supseteq \ldots$ is a descending chain of ideals of L/I. By assumption there exists $m \in \mathbb{N}$ such that $I_m \cap I = I_n \cap I$ and $[L^{(m)}, I_m] \subseteq I_n + I$ for all $n \geq m$. Hence $[L^{(m+1)}, I_m] \subseteq I_n$ and L is quasi-Artinian.

<u>Theorem 3.6</u>

If L_1 and L_2 are quasi-Artinian Lie algebras, then

$L = L_1 \oplus L_2$ is quasi-Artinian.

Proof

Let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of ideals of L. Then $[L_1,I_1] \supseteq [L_1,I_2] \supseteq \ldots$ is a descending chain of ideals of L_1 and $[L_2,I_1] \supseteq [L_2,I_2] \supseteq \ldots$ is a descending chain of ideals of L_2. But L_1 and L_2 are quasi-Artinian, hence there exists $m \in \mathbb{N}$ such that $[L_1^{(m)}, [L_1,I_m]] \subseteq I_n$ and $[L_2^{(m)},[L_2,I_m]] \subseteq I_n$ for all $n \geq m$. Therefore by the Jacobi identity, $[L_1^{(m+1)},I_m] \subseteq I_n$ and $[L_2^{(m+1)},I_m] \subseteq I_n$. Hence $[L^{(m+1)},I_m] \subseteq I_n$. □

Corollary 3.7

A finite direct sum of quasi-Artinian Lie algebras is quasi-Artinian.

Theorem 3.8

Let L be a locally nilpotent quasi-Artinian Lie algebra. Then L is soluble.

Proof

Suppose L is not soluble. Then there is a non-soluble ideal I of L. We claim that I contains a minimal non-soluble ideal of L. Suppose for a contradiction that this is not the case. Let $I = I_1$. Then $0 \neq I_1^{(2)} \subseteq [L^{(1)},I_1]$ and $[L^{(1)},I_1]$ is a non-soluble ideal of L, since I_1 is not soluble. So there is a non-soluble ideal I_2 of L such that $I_2 \subsetneq [L^{(1)},I_1] \subseteq I_1$. Now $0 \neq I_2^{(3)} \subseteq [L^{(2)},I_2]$ and $[L^{(2)},I_2]$ is a non-soluble ideal of L since I_2 is not soluble. So there is a non-soluble ideal I_3 of L such that $I_3 \subsetneq [L^{(2)},I_2] \subseteq I_2$. Continuing this process, there is a non-soluble ideal $I_n \subsetneq [L^{(n-1)},I_{n-1}] \subseteq I_{n-1}$. Then $0 \neq I_n^{(n+1)} \subseteq [L^{(n)},I_n]$ and $[L^{(n)},I_n]$ is a non-soluble ideal of L since I_n is not soluble. So there is a non-soluble ideal I_{n+1} of L such that $I_{n+1} \subsetneq [L^{(n)},I_n] \subseteq I_n$ and so on. Finally the descending chain $I_1 \supset I_2 \supset \ldots$ contradicts the hypothesis that L is quasi-Artinian.

Thus there is such a minimal non-soluble ideal, call it J. But $J^2 \subseteq J$ and J^2 is not soluble, hence $J = J^2$ by the minimality of J. Now either $\zeta_1(J) = 0$ or $\zeta_1(J) \neq 0$.

First, suppose $\zeta_1(J) = 0$. Let $\mathfrak{C} = \{K \triangleleft L \mid K \subseteq J$ and $[K,J] \neq 0\}$. Then $\mathfrak{C} \neq \emptyset$, since $J \in \mathfrak{C}$. We claim that $\mathfrak{C}$ has a minimal element. Suppose not. Put $J = J_1$. Then $0 \neq [J_1^{(2)},J] \subseteq [[L^{(1)},J_1],J]$, so $[L^{(1)},J_1] \in \mathfrak{C}$. Choose $J_2 \in \mathfrak{C}$ such that $J_2 \subsetneq [L^{(1)},J_1] \subseteq J_1$. Then $0 \neq [J,J_2] = [J^2,J_2] \subseteq [J,[J,J_2]] = [J,[J^{(2)},J_2]] \subseteq [J,[L^{(2)},J_2]]$. Hence $[L^{(2)},J_2] \in \mathfrak{C}$ and so on. Choose $J_n \in \mathfrak{C}$ such that $J_n \subsetneq [L^{(n-1)},J_{n-1}] \subseteq J_{n-1}$. Then $0 \neq [J,J_n] = [J^{(n+1)},J_n] \subseteq [J,[L^{(n)},J_n]]$. Therefore $[L^{(n)},J_n] \in \mathfrak{C}$. Repeat this process; then the descending chain of ideals $J_1 \supset J_2 \supset \ldots$ contradicts the hypothesis that L is quasi-Artinian. Thus $\mathfrak{C}$ has a minimal element say K. If K is a minimal ideal of L, then K is central (see [5], Lemma 7.1.6, p. 137) which is a contradiction.

If K is not a minimal ideal of L, then $K \supset H$ and $H \triangleleft L$. Now either $[H,J] = 0$, or $[H,J] \neq 0$. If $[H,J] = 0$, then $H \subseteq C_L(J)$, but $H \subseteq J$, hence $H \subseteq C_L(J) \cap J = \zeta_1(J) = 0$. If $[H,J] \neq 0$, then $H = K$ by minimality of K and K is a minimal ideal of L and in both cases we get a contradiction.

Hence $\zeta_1(J) \neq 0$. Let U be the hypercentre of J. Then $U \triangleleft L$ and $U^{(\alpha)} = 0$ for some infinite α. But L is quasi-Artinian, so by Proposition 3.2, $U^{(\alpha)} = U^{(n)} = 0$ for finite n and so U is soluble. Now J/U is a minimal non-soluble ideal of L/U and $J/U = (J/U)^2$ with $\zeta_1(J/U) = 0$, and a similar argument as above again gives a contradiction. Therefore L is soluble. □

Remark

(1) It appears likely that a theory of prime ideals of quasi-Artinian Lie algebras may exist analogous to that for Min-$\triangleleft$. In particular this would be the case if every semi-simple quasi-Artinian Lie algebra were Artinian. We know no example that disproves this, but it remains an open question.

(2) It is possible to define the notion of quasi-Artinian groups in an analogous way and the proofs of Theorems 3.4, 3.5, 3.6 and 3.3 carry over in this case without difficulties.

CHAPTER FOUR : ON THE JOIN OF SUBIDEALS

It is well-known that the join of two subideals of a Lie algebra need not be a subideal (see Amayo and Stewart [5], Lemma 2.1.11, p. 41). This raises the question of finding conditions under which the join is a subideal. The same question arises in group theory. Wielandt [35, Theorem 2.10.5, p. 41] has shown that: If $\{H_\lambda \mid \lambda \in \Lambda\}$ is a set of subnormal subgroups of a group G and J is their join, then J is subnormal in G if and only if the set of subnormal subgroups of G lying in J contains a maximal member. Following [1] we shall obtain a similar result for Lie algebras. We also find another condition under which the join of subideals is a subideal by imposing conditions on the circle product of subideals. Our main results are as follows:

Let L be a Lie algebra over a field of characteristic zero and let H_λ si L, $\lambda \in \Lambda$. Suppose that $J = \langle H_\lambda \mid \lambda \in \Lambda \rangle$ and $\mathbb{B} = \{B \mid B < J,\ B \text{ si } L\}$. Then J si L if and only if $\mathbb{B}$ has a maximal element (Theorem 4. 4).

Suppose that $\mathfrak{X}$ is an $\{I, N_0\}$-closed and locally coalescent class over any field. Let H and K be $\mathfrak{X}$-subideals of a Lie algebra L with $J = \langle H,K \rangle$. If $H \circ K/(H \circ K)^2$ is finitely generated then J si L and $J \in \mathfrak{X}$ (Theorem 4.12).

We start with a construction of Lie algebras of power series used in Amayo and Stewart [5, pp. 77, 78, 80].

Let L be a Lie algebra over a field F of characteristic zero and let $F_0 = F\langle t\rangle$ be the field of formal power series in the intermediate t. Let $L^\dagger$ be the set of all formal power series $x = \sum_{r=n}^{\infty} x_r t^r$, $x_r \in L$ and $n = n(x) \in \mathbb{Z}$. Let $y = \Sigma y_r t^r \in L^\dagger$ and define addition, multiplication and multiplication of elements of $L^\dagger$ by scalars from F_0 according to the rules:

$$x + y = \Sigma(x_r + y_r)t^r,$$

$$[x,y] = \Sigma z_r t^r, \text{ where } z_r = \sum_{i+j=r} [x_i, y_j]$$

$$\alpha x = \Sigma c_r t^r, \text{ where } c_r = \sum_{i+j=r} \alpha_i x_j.$$

It is easy to verify that this makes $L^\dagger$ into a Lie algebra over F_0. Let $M \leq L$ and $M^\dagger$ be the set of all elements $x = \Sigma x_r t^r \in L^\dagger$ with $x_r \in M$. Then clearly $M^\dagger$ is F_0-subalgebra of $L^\dagger$ and it is easy to prove the following which is Amayo and Stewart [5, Lemma 4.1.1(b) p.79].

<u>Lemma</u> 4.1

Let L be a Lie algebra over a field F of characteristic zero and let $K \triangleleft H \leq L$. Then $K^\dagger \triangleleft H^\dagger \leq L^\dagger$. In particular if $H \triangleleft^m L$, then $H^\dagger \triangleleft^m L^\dagger$. □

Let L be a Lie algebra over F and let $x \neq 0$, $x \in L^\dagger$. Then x can be written uniquely in the form $x = t^m \sum_{r=0}^{\infty} x_r t^r$, where $m = m(x) \in \mathbb{Z}$ and $x_0 \neq 0$ is the first non-zero coefficient of x. x_0 is called the *first coefficient* of x. Clearly every non-zero element $\alpha \in F_0$ has a similar expression: $\alpha = t^s \Sigma_{r=0}^{\infty} \alpha_r t^r$, $\alpha_0 \neq 0$ is called the *first coefficient* of α. Let $y \neq 0$, $y \in L^{\uparrow}$ so that $y = t^n \Sigma_{r=0}^{\infty} y_r t^r$ where $y_0 \neq 0$. Clearly $[x,y] = t^{m+n}[x_0,y_0] + t^{m+n} \Sigma_{r=1}^{\infty}(\Sigma_{i+j=r}[x_i,y_j])t^r$, and for any $\alpha,\beta \in F$, $\alpha x + t^{m-n} \beta y = t^m \Sigma_{r=0}^{\infty}(\alpha x_r + \beta y_r)t^r$. Now let M be a subset of $L^{\uparrow}$ and let $M^{\downarrow} = \{x \in L \mid x = 0$ or x is the first coefficient of some element of M$\}$. Then the above equations lead to the following result (see Amayo and Stewart [5, Lemma 4.1.2 (a), (b), (c), (f), p. 80]).

Lemma 4.2

Let L be a Lie algebra over a field F of characteristic zero. Then

(a) If M is a subspace (resp. subalgebra) of $L^{\uparrow}$, then $M^{\downarrow}$ is a subspace (resp. subalgebra) of L.

(b) If $N \triangleleft M < L^{\uparrow}$, then $N^{\downarrow} \triangleleft M^{\downarrow} < L$. In particular if $N \triangleleft^n L^{\uparrow}$, then $N^{\downarrow} \triangleleft^n L$.

(c) Let M and N be subsets of $L^{\uparrow}$. Then $[M^{\downarrow},N^{\downarrow}] \subseteq [M,N]^{\downarrow}$ $(M \cap N)^{\downarrow} \subseteq M^{\downarrow} \cap N^{\downarrow}$; $M^{\downarrow} + N^{\downarrow} \subseteq (M + N)^{\downarrow}$

(d) If M is a subspace of L, then $M^{\uparrow\downarrow} = M$. □

Now we prove the following which is the Lie algebra analogue of Wielandt [35, Lemma 2.10.4, p. 40].

Lemma 4.3

Let L be a Lie algebra over a field of characteristic zero and let $S < L$. Let $\mathbb{B} = \{B \mid B < S,\ B \text{ si } L\}$ and let H be a maximal element of $\mathbb{B}$. Then $H \lhd S$ and $H > B$ for every $B \in \mathbb{B}$.

Proof

Let H have subideal index m in S and suppose that $m \geq 2$. Denote the i^{th} ideal closure of H in S by H_i. It follows that there exists $x \in H_{m-2}$ with $[H,x] \not\subseteq H$. By Lemma 4.1, $H^{\uparrow} \lhd^m L^{\uparrow}$ and $H^{\uparrow} < S^{\uparrow}$. Let $\theta = \exp(t \text{ ad } x)$. Then $H^{\uparrow\theta}$ si $L^{\uparrow}$ and $H^{\uparrow\theta} \subseteq H^{\uparrow}_{m-1}$. But $H^{\uparrow} \lhd H^{\uparrow}_{m-1}$, hence $H^{\uparrow\theta}$ idealises $H^{\uparrow}$ and $H^{\uparrow} + H^{\uparrow\theta}$ si $L^{\uparrow}$ and $H^{\uparrow} + H^{\uparrow\theta} < S^{\uparrow}$. By Lemma 4.2, we have $(H^{\uparrow} + H^{\uparrow\theta})^{\downarrow}$ si L and $(H^{\uparrow} + H^{\uparrow\theta})^{\downarrow} < S$. Hence $(H^{\uparrow} + H^{\uparrow\theta})^{\downarrow} \in \mathbb{B}$. By Lemma 4.2, $H^{\uparrow\downarrow} = H$ and $H \subseteq (H^{\uparrow} + H^{\uparrow\theta})^{\downarrow}$, but H is maximal, hence $(H^{\uparrow} + H^{\uparrow\theta})^{\downarrow} = H$. Now take $h \in H$ such that $[h,x] \notin H$. Then $h^{\theta} - h = [h,x]t + \ldots$ and therefore $h^{\theta} - h \in (H^{\uparrow} + H^{\uparrow\theta})^{\downarrow} = H$ which is a contradiction. Therefore $H \lhd S$ and clearly $H > B$ for every $B \in \mathbb{B}$. □

Theorem 4.4

Let L be a Lie algebra over a field of characteristic zero and H_λ si L, $\lambda \in \Lambda$. Let $J = \langle H_\lambda \mid \lambda \in \Lambda \rangle$ and $\mathbb{B} = \{B \mid B < J,\ B \text{ si } L\}$. Then J si L if and only if $\mathbb{B}$ has a maximal element.

Proof

The only if part is clear. To prove the if part, let H be a maximal element in $\mathbb{B}$. Then by Lemma 4.3, $H \lhd J$ and each $H_\lambda < H$ so $J < H$ and $J = H$. Therefore J si L.

Corollary 4.5

Let L be a Lie algebra over a field of characteristic zero and let $J = \langle H | H \text{ si } L, \lambda \in \Lambda \rangle$. Then J si L if one of the following holds.

(i) $L \in$ max-si

(ii) $J \in$ max-si

Proof

Let $\mathbb{B} = \{B | B < J, B \text{ si } L\}$. By assumption (i) or (ii) $\mathbb{B}$ has a maximal element. Therefore by Theorem 4.4, J si L. □.

Next we prove the following, which is the Lie algebra analogue of a well-known result in group theory.

Theorem 4.6

Let L be a Lie algebra over a field of characteristic zero and let H, K be subideals of L. Suppose that the set of subideals of L lying between H and $J = \langle H,K \rangle$ contains at least one maximal member. Then J si L.

To prove this we need the following well-known lemma:

Lemma 4.7

Let L be a Lie algebra and suppose that $H \triangleleft^m L$, $K \triangleleft^n L$ and $J = \langle H,K \rangle$. If $H \triangleleft J$ then $J \triangleleft^{mn} L$.

Proof

See Amayo and Stewart [5, Lemma 2.1.2, p. 33]. □

Proof of Theorem 4.6

Without loss of generality we may assume that H is a maximal member of the set of subideals of L lying in J and containing the original H. By Lemma 4.3, $H \triangleleft J$ and by Lemma 4.7, H si L. □

As an application of Theorem 4.6, we have the following, which is due to Hartley [13]. It is proved in Amayo and Stewart [5, p.64] by a different method.

Corollary 4.8

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H, K be subideals of L. Then $J = \langle H,K \rangle$ is a finite-dimensional subideal of L.

Proof

That J si L follows from Theorem 4.6. Further J is finite-dimensional since L is. □

Corollary 4.9

Let L be a Lie algebra over a field of characteristic zero and let H, K be subideals of L. If H has finite-codimension in J = <H,K>, then J si L.

Proof

Since H has finite codimension in J, it follows that the set of subideals of L lying between H and J has a maximal element and by Theorem 4.6, J si L. □

Finally we find another condition under which the join of two subideals of a Lie algebra is a subideal. For this we shall need the following definitions (see Amayo and Stewart [5, pp. 18-20, 30, 67]). A class $\mathfrak{X}$ is I-*closed* provided every subideal of an $\mathfrak{X}$-algebra is always an $\mathfrak{X}$-algebra. A class $\mathfrak{X}$ is N_0-*closed* if whenever $H, K \triangleleft L$ and $H,K \in \mathfrak{X}$, then $H + K \in \mathfrak{X}$. A class $\mathfrak{X}$ is *locally coalescent* if and only if whenever H and K are $\mathfrak{X}$-subideals of a Lie algebra L, then to every finitely-generated subalgebra C of J = <H,K> there corresponds an $\mathfrak{X}$-subideal X of L such that $C \leq X \leq J$.

Let H and K be a subset of a Lie algebra L. The *circle product* of H and K denoted by $H \circ K$, is defined as $H \circ K = [H,K]^{H \cup K}$. It is clear that $H \circ K$ is the smallest ideal of J = <H,K> containing [H,K].

Now we prove the following, which is the Lie algebra analogue of Robinson [24, Lemma 2.3, p. 149].

<u>Proposition 4.10</u>

Let L be a Lie algebra over any field and let H, K be subideals of L with $J = \langle H,K\rangle$. Then the following are equivalent:

(i) J si L

(ii) H^K si L

(iii) $H \circ K$ si L.

<u>Proof</u>

(i) $\Rightarrow$ (iii). Since $H^K = H + H \circ K \lhd J$ and J si L, it follows that H^K si L.

(ii) $\Rightarrow$ (iii). From $H \circ K \lhd J$, we have $H \circ K \lhd H^K$. But H^K si L hence $H \circ K$ si L.

(iii) $\Rightarrow$ (i). Let $H \circ K$ si L. Then since H si L and H idealises $H \circ K$, it follows from Lemma 4.7, that H^K si L. However K idealises H^K so a further application of Lemma 4.7, shows $H^K + K = J$ si L. □

<u>Corollary 4.11</u>

Let $L \in \mathfrak{N}\mathfrak{A}$ and H,K are subideals of L, then $J = \langle H,K\rangle$ si L.

<u>Proof</u>

Since $L \in \mathfrak{N}\mathfrak{A}$, there exists $N \lhd L$ with $N \in \mathfrak{N}$ and $L/N \in \mathfrak{A}$. Hence $H \circ K \subseteq N$ and so $H \circ K$ si $N \lhd L$. Therefore $H \circ K$ si L and by Proposition 4.10, J si L. □

Theorem 4.12

Suppose that $\mathfrak{X}$ is an $\{I, N_0\}$-closed and locally coalescent class over any field. Let H and K be $\mathfrak{X}$-subideals of a Lie algebra L with $J = \langle H,K \rangle$. If $H \circ K/(H \circ K)^2$ is finitely-generated, then J si L and $J \in \mathfrak{X}$.

To prove this we need the following well-known results.

Lemma 4.13

Let H and K be subalgebras of a Lie algebra L such that $L = H + K^2$. Then $L = H + K^{n+1}$ for any $n \in \mathbb{N}$. If in addition $H \cap K$ si K then for any $n \in \mathbb{N}$, $L = H + K^{(n)}$.

Proof

See Amayo and Stewart [5, Lemma 2.1.9, p. 40]. □

Lemma 4.14

Let L be a Lie algebra and let $H_1 \triangleleft^{h_1} L$, $H_2 \triangleleft^{h_2} L$ and $J = \langle H_1, H_2 \rangle$. Then there exists $\lambda_3 = \lambda_3(h,r)$ such that $J^{(\lambda_3)} \leq H_1^{(r_1)} + H_2^{(r_2)}$ and $J^{(\lambda_3)} \triangleleft^{\lambda_3} L$ whenever $h_1 + h_2 \leq h$ and $r_1 + r_2 \leq r$.

Proof

See Amayo and Stewart [5, Theorem 2.2.7, p. 48]. □

Lemma 4.15

Let $\mathfrak{X}$ be an $\{I, N_0\}$-closed class of Lie algebras and suppose that H and K are $\mathfrak{X}$-subideals of a Lie algebra L and $J = \langle H,K\rangle$. If H and K are permutable then J is an $\mathfrak{X}$-subideal of L.

Proof

See Amayo and Stewart [5, Theorem 2.2.13, p. 54]. □

Lemma 4.16

Let L be a Lie algebra and $J = \langle H_1,\dots,H_n\rangle$ with $H_i \lhd^{h_i} L$ for $i = 1,\dots,n$. If each H_i lies in an $\{I,N_0\}$-closed class $\mathfrak{X}$ then $J^{(\lambda)} \in \mathfrak{X}$ and so $J \in \mathfrak{X}\mathfrak{A}^{\lambda}$.

Proof

See Amayo and Stewart [5, Corollary 2.2.17 , p. 57]. □

Proof of Theorem 4.12

Let $M = H \circ K$. Now there exists a finitely-generated subalgebra C of M such that $M = C + M^2$. By the local coalescence of $\mathfrak{X}$ there exists an $\mathfrak{X}$-subideal X of L with $C \leq X \leq J$. Thus if $N = X \cap M$, then $N \lhd X$ si L and so N si L, $N \in I\mathfrak{X} = \mathfrak{X}$ and $M = N + M^2$. From Lemma 4.13, we have $M = N + M^{(r)}$ for all r. By Lemma 4.16, we have $J^{(r)} \in \mathfrak{X}$ for some r and so $M^{(r)} \in I\mathfrak{X} = \mathfrak{X}$. Finally by Lemma 4.15, we have $M = N + M^{(r)} \in \mathfrak{X}$ and M si L (for $J^{(r)}$ and so $M^{(r)}$ si L for some r by Lemma 4.14). We also have by Lemma 4.15, that $H + M$, $K + M \in \mathfrak{X}$ and $H + M$, $K + M$ si L and so by the same result $J = H + M + K + M \in \mathfrak{X}$

and J si L. □

<u>Corollary 4.17</u>

Let L be a Lie algebra over a field of characteristic zero and let H, K be subideals of L. If $H \circ K \in$ Max-si or $H \circ K \in$ Min-si, then $J = \langle H,K \rangle$ si L.

<u>Proof</u>

The hypothesis of Theorem 4.12 are satisfied for these two classes. □

CHAPTER FIVE : SUBIDEALS AND ASCENDANT SUBALGEBRAS OF LIE ALGEBRAS

Wielandt [34] has shown that a subgroup H of a finite group G is subnormal in G if and only if for each $g \in G$ and $h \in H$ there exists an integer n such that $[g,_n h] \in H$. This, and related criteria given by Wielandt in the same paper, have been extended to various classes of infinite groups by Peng [22, 23], Hartley and Peng [14], Whitehead [32, 33] and Wehrfritz [31]. The Lie algebra analogue of Wieland's criteria have been investigated in a various class of Lie algebras by Chao and Stitzinger [8], Kawamoto [17], Stewart [27], Stitzinger [28], Tôgô [29], Tôgô, Honda and Sakomoto [30].

Following [4], we give some criteria for subideality and ascendancy in Lie algebras, similar to Wielandt's stated in terms of the circle product. The main results are as follows:

If L is a finite-dimensional Lie algebra over a field of characteristic zero and H is a subalgebra of L, then H si L if and only if one of the following conditions holds:

(i) for each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x\rangle \circ \langle h_1\rangle \circ \langle h_2\rangle \circ \dots \circ \langle h_n\rangle \subseteq H$ for all $h_1,\dots,h_n \in H$.

(ii) for each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x\rangle \circ_n \langle h\rangle \subseteq H$.

(iii) for each $h \in H$ there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x\rangle \circ_n \langle h\rangle \subseteq H$. (Theorem 5.2.1).

A generalization to infinite-dimensional Lie algebras leads to the following.

Let L be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H < L$.

(i) If for each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x\rangle \circ \langle h_1\rangle \circ \langle h_2\rangle \circ \ldots \langle h_n\rangle \subseteq H$ for any $h_1,\ldots,h_n \in H$, then H asc L.

(ii) suppose that H is finite-dimensional

(a) If for each $h \in H$, there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x\rangle \circ_n \langle h\rangle \subseteq H$, then H si L.

(b) If for each $x \in L$ and $h \in H$, there exists an integer $n = n(x,h)$ such that $\langle x\rangle \circ_n \langle h\rangle \subseteq H$, then H asc L. (Theorem 5.3.6).

Finally if L is an ideally finite Lie algebra (see below) over a field of characteristic zero and $H < L$, then H asc L if either of the following conditions holds:

(i) For each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x\rangle \circ_n \langle h\rangle \subseteq H$.

(ii) H asc $\langle H,x\rangle$ for each $x \in L$. (Theorem 5.3.7)

5.1 Definitions and basic results

Let L be a Lie algebra over any field. The *Fitting radical* $\nu(L)$ is the sum of nilpotent ideals of L (equal to the nil radical in finite dimensions). If L is finite-dimensional over a field of characteristic zero, then $\nu(L)$ contains every nilpotent subideal of L (see Amayo and Stewart [5], p. 114). A finite-dimensional Lie algebra L is said to be *split* (see Jacobson [16], p. 108), if the characteristic roots of every $ad_L h$, $h \in H$, where H is a Cartan subalgebra L, are in the base field. The *Hirsch-Plotkin radical* $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. If the underlying field has characteristic zero we define the *Gruenberg radical* $\gamma(L)$ to be the subalgebra generated by the nilpotent ascendant subalgebras of L (see Amayo and Stewart [5], pp. 113, 114). An algebra is *locally finite* if every finite set of elements is contained in a finite-dimensional subalgebra. If H si L, then $H^{\omega} = \bigcap_{n=1}^{\infty} H^n \triangleleft L$, where ω is the first infinite-ordinal (see Amayo and Stewart [5], Lemma 1.3.2, p. 10). Let A and B be subsets of a Lie algebra L. The *circle product* $A \circ B$ of A and B is defined (see Amayo and Stewart [5], p. 30), as $A \circ B = [A,B]^{A \cup B}$. We also let

$A \circ_1 B = A \circ B$ and define recursively $A \circ_{m+1} B = (A \circ_m B) \circ B$ for all positive integers m. We say that $x \in L$ is a *left Engel element* (see [5], p. 339) if for each $y \in L$ we can find $n = n(x,y) \geq 0$ such that $[y,_n x] = 0$. If n can be chosen independently of y we say that x is a *bounded left Engel element*. The sets of left Engel, bounded left Engel elements are denoted by $\mathfrak{e}(L)$ and $\bar{\mathfrak{e}}(L)$ respectively. A *local system* for a Lie algebra L (see Stewart [26], p. 33) is a collection $\{L_i\}_{i \in I}$ of subalgebras of L which generate L and have the property that whenever $i,j \in I$ there exists $m \in I$ such that $\langle L_i, L_j \rangle \leq L_m$. A Lie algebra is said to be *ideally finite* (see [26], p. 34) if it has a local system of finite-dimensional ideals. A *Fitting class* is (see Amayo and Stewart [5], p. 259) a subclass $\mathfrak{X}$ of $\mathfrak{F}$ which is $\{N_0, I\}$-closed. A subalgebra H of a Lie algebra L is called *serial*, written H ser L, if there is a series from H to L, (see [5], p. 258). We write $L \in E'\mathfrak{A}$, see [5], p. 28) if L has an ascending abelian series $(L_\alpha)_{\alpha<\lambda}$. If each $L_\alpha (\alpha < \lambda)$ is an ideal of L, then $L \in E'(\triangleleft)\mathfrak{A}$, L is *hyperabelian*. Let L be a Lie algebra over a field F. A *universal enveloping algebra* of L is a pair (U,i), where U is an associative algebra with 1 over F, $i: L \longrightarrow U$ is a linear map satisfying

$$i([x,y]) = i(x)i(y) - i(y)i(x) \text{ for } x,y \in L \tag{1}$$

and the following holds: for any associative F-algebra A with 1 and any linear map $j: L \longrightarrow A$ satisfying (1) there exists a unique homomorphism of algebras $\theta : U \longrightarrow A$ (sending 1 to 1) such that $\theta \circ i = j$ (see Humphreys [15], p. 90). We say that x *acts nilpotently* on L if $[L,_n x] = 0$ for some $n \in \mathbb{N}$.

Let H be an L-module. We say that H *locally algebraic* if for each $x \in L$ and $a \in H$ there exists a polynomial $P = P_{a,x}$ such that $ap(x) = 0$. Note that Curtis [9] defines the concept of a locally algebraic transformation as follows:

Let V be a vector space over a perfect field F. A linear transformation α is *locally algebraic* if every vector $x \in V$ is contained in a finite-dimensional α-invariant subspace of V. α is *algebraic* if $P(\alpha) = 0$, where P is a non-zero polynomial with coefficients in F. Now we have the following:

<u>Lemma 5.1.1</u>

An L-module is locally algebraic in our sense if and only if the transformations induced on it by the L-action are locally algebraic in Curtis's sense.

<u>Proof</u>

Let A be a locally algebraic L-module in our sense and let $a \in A$. Suppose that $\langle a \rangle_\alpha$ is an α-invariant subspace of A. We claim that $\langle a \rangle_\alpha$ is finite-dimensional. By assumption we have $ap(\alpha) = 0$ for some non-zero polynomial P. Therefore $a\alpha^m = \sum_{i=0}^{m-1} \lambda_i \, a\alpha^i$ where $\lambda_i \in F$.

Therefore by induction $a\alpha^n = \sum_{i=1}^{m-1} \mu_i \, a\alpha^i$ for all $n \geq m$. So $\langle a\rangle_\alpha \subseteq \langle a, a\alpha, \ldots, a\alpha^{m-1}\rangle$ and $\langle a\rangle_\alpha$ is finite-dimensional.

The converse is clear. □

Finally we define three new relations ci, sci, nci as follows: Let $H \leq L$. Write H ci L if for each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x\rangle \circ_n \langle h\rangle \subseteq H$. H sci L if for each $h \in H$ there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x\rangle \circ_n \langle h\rangle \subseteq H$.

H nci L if for each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x\rangle \circ \langle h_1\rangle \circ \langle h_2\rangle \circ \ldots \circ \langle h_n\rangle \subseteq H$ for any $h_1, h_2, \ldots, h_n \in H$.

The following implications are clear.

$$\begin{array}{ccccc}
H \triangleleft^{\omega} L & \longleftarrow & H \text{ si } L & \longrightarrow & H \text{ nci } L \\
\downarrow & & \downarrow & & \downarrow \\
H \text{ ci } L & \longleftarrow & H \text{ sci } L & \longrightarrow & H \text{ ci } L
\end{array}$$

The object is to produce partial converses of these implications.

Next we prove the following:

<u>Lemma 5.1.2</u>

Let L be a Lie algebra over any field and let $H \leq L$. Let Δ be any of the relations ci, sci, nci. Then

(i) If $K < L$, $H \Delta L$ then $H \cap K \Delta K$.

(ii) If f is a homomorphism of L onto $\bar{L}$ and $H \Delta L$, then $f(H) \Delta \bar{L}$. If $\bar{H} \Delta \bar{L}$, then $f^{-1}(\bar{H}) \Delta L$.

(iii) K si $H \Delta L$ implies $K \Delta L$.

<u>Proof</u>

(i) is clear.

(ii) This follows from the fact that if H and K are subalgebras of L and f is a homeomorphism of L onto $\bar{L}$, then

$$f(H \circ K) = f(H) \circ f(K) \quad \text{and} \quad f^{-1}(\bar{H} \circ \bar{K}) = f^{-1}(\bar{H}) \circ f^{-1}(\bar{K}).$$

(iii) Suppose $K \triangleleft^m H$. We prove by induction on m that if $K \triangleleft^m H$ and H ci L, then K ci L. Suppose that $m = 1$. Then since for each $x \in L$ and $y \in K$ it follows that $x \in L$ and $y \in H$, so there exists an integer $n = n(x,y)$ such that $\langle x \rangle \circ_n \langle y \rangle \subseteq H$. But $K \triangleleft H$, hence $\langle x \rangle \circ_n \langle y \rangle \circ \langle y \rangle \subseteq K$. Therefore $\langle x \rangle \circ_{n+1} \langle y \rangle \subseteq K$ and K ci L. Now assume that the result is true for some $i \geq 1$. Then $K \triangleleft^i H$ ci L implies K ci L. So if $K \triangleleft^{i+1} H$ ci L we have $K \triangleleft K_i \triangleleft^i H$ ci L and by induction K_i ci L. Therefore as in the case $m = 1$, K ci L.

By a similar argument we can show that if Δ is either of the remaining relations sci or nci, and K si $H \Delta L$, then $K \Delta L$. $\square$

5.2 Finite dimensions

Theorem 5.2.1

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let $H < L$. Then the following are equivalent.

(i) H si L

(ii) H nci L

(iii) H ci L

(iv) H sci L

To prove Theorem 5.2.1, we need the following well-known results.

Lemma 5.2.2

If L is a nilpotent Lie algebra of class c, and if $H < L$, then $H \lhd^{c} L$.

Proof

See Amayo and Stewart [5, Lemma 1.3.7, p. 12]. □

Lemma 5.2.3

Let L be a Lie algebra over a field of characteristic zero. If $L \in E'\mathfrak{A}$, then $\rho(L) \subseteq \varepsilon(L) = \gamma(L)$.

Proof

See Amayo and Stewart [5, Theorem 16.4.2(a), p. 341]. □

Lemma 5.2.4

If L is a finite-dimensional semi-simple Lie algebra over a field of characteristic zero, then every finite-dimensional module for L is completely reducible.

Proof

See Jacobson [16, Theorem 8, p. 79]. □

Lemma 5.2.5

Let L be a Lie algebra of linear transformations in a finite-dimensional vector space V over a field of characterisitc zero. Assume that L is completely reducible. Then every non-zero nilpotent element of L can be imbedded in a three-dimensional split simple subalgebra of L.

Proof

See Jacobson [16, Theorem 17(1), p. 100]. □

Lemma 5.2.6 (Engel's Theorem)

If L is a finite-dimensional Lie algebra, then L is nilpotent if and only if ad x is nilpotent for every $x \in L$.

Proof

See Humphreys [15, p. 12]. □

<u>Proof of Theorem 5.2.1</u>

Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii), so we only need to prove that (iii) $\Rightarrow$ (i). For any $h \in H$, $\mathrm{ad}_L h$ induces a linear transformation $\alpha(h)$ of the space L/H. By assumption each $\alpha(h)$ is nil. Since the space L/H is finite-dimensional, $\alpha(h)$ is nilpotent. Therefore by Lemma 5.2.6 $\alpha(H)$ is nilpotent. Hence there exists an integer m such that $\alpha(h_1)\cdot\alpha(h_2) \ldots \alpha(h_m) = 0$ for any $h_1,h_2,\ldots,h_m \in H$. Therefore $[L,_m H] \subseteq H$. So for all $n \in \mathbb{N}$ $[L,H^{n+m}] \subseteq H^{n+1}$ from which it follows that $H^\omega \lhd L$. Now we argue for a contradiction assuming that L is a counterexample of minimal dimension. We have H ci L but H is not a subideal of L. If $H^\omega \neq 0$, then by minimality H/H^ω si L/H^ω, and H si L, a contradiction. Therefore $H^\omega = 0$ and H is nilpotent. Hence if $h \in H$, then $\langle h\rangle$ si H by Lemma 5.2.2. If the theorem were true for the case dim H = 1, it would follow that $\langle h\rangle$ si L for all $h \in H$, hence $H \leq \nu(L)$. Therefore H would be a subideal. It follows that we may assume that dim H = 1, so that $H = \langle h\rangle$ for some $h \in L$. For all $x \in L$ we have $[x,_m h] = 0$ for some $m > 0$. Since L has finite dimension $[L,_m h] = 0$ for some $m > 0$. Let $S = \sigma(L)$. If $S = L$, then every element h for which $\mathrm{ad}_L h$ is nilpotent lies in $\nu(L)$ by Lemma 5.2.3. Therefore $H \leq \nu(L)$, so H si L, which again is a contradiction. Hence $S \neq L$. It follows that $S + H \neq L$, since $S + H$ is soluble. By minimality we have H si S + H.

If $S \neq 0$, then $(S + H)/S$ si L/S by minimality, which implies that H si L. Therefore $S = 0$ and L is semi-simple. By Lemma 5.2.4 and Lemma 5.2.5, there is an element $k \in L$ such that $\langle h,k \rangle$ is a three-dimensional split simple Lie algebra. By Lemma 5.1.2, $\langle h \rangle$ ci $\langle h,k \rangle = T$. Therefore $\langle k \rangle \circ_n \langle h \rangle \subseteq \langle h \rangle$, but $\langle k \rangle \circ_n \langle h \rangle = T$ hence $\langle h \rangle = T$ which is a contradiction, and this completes the proof. □

As a consequence of Theorem 5.2.1, we have the following, which is Stewart [27, Theorem 1].

Corollary 5.2.7

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let $H \in L$. If H si $\langle H,x \rangle$ for all $x \in L$, then H si L.

Proof

Let $K = \langle H,x \rangle$, for some $x \in L$, then by hypothesis, we have $K \circ_n H < H$ for some n. Hence H ci L and by Theorem 5.2.1, we have H si L. □

Remark

The proof of Theorem 5.2.1 fails in characteristic $P > 0$. However we have the following.

Proposition 5.2.8

Let L be a finite-dimensional soluble Lie algebra over any field and let $H < L$. If H ci L, then H si L.

Proof

This follows from Tôgô [29, Theorem 8 and Corollary (b) of Theorem 2]. □

5.3 Infinite-dimensions

For infinite dimensional Lie algebras we do not expect a result like Theorem 5.2.1 in general. For example let L be the Lie algebra of Amayo and Stewart [5, p. 119], that is $L = X + \langle\sigma\rangle$ where X is an abelian Lie algebra with basis $x_0, x_1, x_2, \ldots$ and σ is a derivation on X defined by $x_0\sigma = 0$, $x_i\sigma = x_{i-1}$ $(i > 0)$. Let $H = \langle\sigma\rangle$. Then since L is locally nilpotent, it follows that $\langle x,\sigma\rangle$ is nilpotent for each $x \in L$. Therefore H ci L, but H is not a subideal of L. (However, H *is* ascendant in L).

Moreover, let F be any field of characteristic zero, and let $A = F[x_1, x_2, \ldots]$ be the polynomial algebra in a countably infinite set of indeterminates x_n over F. Let I be the ideal of A generated by $x_1, x_2^2, \ldots, x_i^i, \ldots$. Considered as an abelian Lie algebra, $P = A/I$ has derivations: $\delta_i : f \longrightarrow f x_i$, $i = 1,2,\ldots$ for each $f \in P$. Then $\delta_i^i = 0$ and $\delta_i\delta_j = \delta_j\delta_i$ for all i,j. Let $H = \langle\delta_i\rangle$ and form the split extension $L = P \dotplus H$. Then L is soluble of derived length two and is locally nilpotent; for each $h \in H$, there exists $n = n(h)$ such that for all $x \in L$ we have $\langle x\rangle \circ_n \langle h\rangle \subseteq H$. Therefore H sci L, but $H = I_L(H)$, so H is neither a subideal nor ascendant. However we have the following:

Proposition 5.3.1

Let L be a Lie algebra over a field F such that $L = A + H$, where A is an abelian ideal of L and $H < L$. If H ci L, then H si L if either of the following conditions holds:

(i) H has finite codimension.

(ii) The characteristic of F is zero and A is finite-dimensional.

Proof

(i) For any $h \in H$, $\mathrm{ad}_L h$ induces a linear transformation $\alpha(h)$ of the space L/H. By assumption each $\alpha(h)$ is nil. But the space L/H is finite-dimensional, hence $\alpha(h)$ is nilpotent. Therefore by Lemma 5.2.6, $\alpha(H)$ is nilpotent. Hence there exists an integer m such that $\alpha(h_1) \ldots \alpha(h_m) = 0$ for any $h_1, h_2, \ldots, h_m \in H$. Thus $[L,{}_mH] \subseteq H$. Now since $A \cap H \lhd L$, we may assume that $A \cap H = 0$. It follows that $[A,{}_mL] = [A,{}_mH] = 0$. Therefore $A \subseteq \zeta_m(L)$ and $H \lhd^m L$.

(ii) This follows from Theorem 5.2.1 applied to $L/C_H(A)$. □

In the rest of this section we shall find certain conditions under which $H \,\Delta\, L$ (where Δ is one of the relations ci, sci, nci) implies that H si L or H asc L.

We start with the following.

Theorem 5.3.2

Let L be a hyperabelian Lie algebra over any field, and let $H < L$.

(i) If H nci L, then H asc L

(ii) If H has finite dimension, then H sci L implies H si L while H ci L implies H asc L.

To prove this we need the following results:

Lemma 5.3.3

Let L be a Lie algebra of algebraic linear transformations of a vector space V over a field F, and let U(L) be the universal enveloping algebra of L. Then L is locally finite if and only if U(L) is locally finite.

Proof

See Curtis [9, Lemma 3.1, p. 456].

Lemma 5.3.4

Let L be a Lie algebra over any field and let $H < L$. Then if $x \in H$ implies $\langle x \rangle$ si L, then every finitely generated subalgebra of H is a subideal of L.

Proof

See Amayo and Stewart [5, Theorem 7.1.5(c), p. 136]. □

<u>Lemma 5.3.5</u>

Let L be a Lie algebra over any field such that $L = A + H$, where A is an abelian ideal of L and $H < L$. Then

(i) If H nci L, then H asc L.

(ii) If H has finite dimension, then

(a) H sci L implies H si L

(b) H ci L implies H asc L.

<u>Proof</u>

(i) Since $A \cap H \triangleleft L$, we may assume that $A \cap H = 0$. Let $x_1,x_2,\ldots,x_n$ be any elements of L. Then $x_i = a_i + h_i$, for some $a_i \in A$ and $h_i \in H$. Now let $a \in A$, then $[a,x_1] = [a,h_1]$ for A is abelian. Hence $[a,x_1,x_2,\ldots,x_n] = [a,h_1,h_2,\ldots,h_n]$. But $[a,h_1,h_2,\ldots,h_n] \in A \cap H = 0$, hence $[a,x_1,x_2,\ldots,x_n] = 0$. Therefore $a \in \zeta_n(L)$. It follows that $A \subseteq \zeta_\omega(L)$. Hence H asc L.

(ii) (a) Since $A \cap H \triangleleft L$, we may assume that $A \cap H = 0$. It follows that for each $h \in H$ there exists $n = n(h)$ such that for all $a \in A$ we have $[a,{}_nh] = 0$. Consider A as an H-module, then A is an algebraic H-module (because $\mathrm{ad}_A h$ is algebraic). Let E be the associative algebra generated by all $\{\mathrm{ad}_A h \mid h \in H\} \cup \{1_A\}$. Then Lemma 5.3.3 implies that E is finite-dimensional. Now if $a \in A$, then $\langle a\rangle_H = aE = \{ae \mid e \in E\}$, where $\langle a \rangle_H$ is the H-module generated by the element a. Hence A is a locally finite H-module. Now if B is any finite-dimensional H-submodule of A, then for each $h \in H$ there exists $n = n(h)$ such that $Bh^n = 0$. Hence by Lemma 5.2.6, we can find $m = m(B)$ such that $BH^m = 0$. It follows that $A \subseteq \zeta_\omega(A + H)$, and $H^\omega \lhd L$. So we may assume that $H^\omega = 0$ and H is nilpotent. By assumption we have $h \in \bar{\mathfrak{e}}(L)$ for each $h \in H$. Therefore $\langle h \rangle$ si L. Hence by Lemma 5.3.4, H si L.

(b) Since $A \cap H \lhd L$, we may assume that $A \cap H = 0$. It follows that for each $a \in A$ and $h \in H$, there exists $n = n(x,h)$ such that $[a,{}_nh] = 0$. Consider A as an H-module, then A is locally algebraic and the argument of Lemma 5.3.3, implies that A is locally finite. Now if B is any finite-dimensional H-submodule of A, then there exists $n = (B,h)$ such that $Bh^n = 0$. Hence by Lemma 5.2.6 we can find $m = m(B)$ such that $BH^m = 0$. It follows that $A \subseteq \zeta_\omega(A + H)$ and H asc L. □

<u>Proof of Theorem 5.3.2</u>

(i) Let $(L_\alpha)_{\alpha<\lambda}$ be an ascending abelian series of ideals of L. Now $(H + L_{\alpha+1})/L_\alpha = (H + L_\alpha)/L_\alpha + L_{\alpha+1}/L_\alpha$ and $L_{\alpha+1}/L_\alpha$ is an abelian ideal of L/L_α. Therefore $L_{\alpha+1}/L_\alpha \lhd (H + L_{\alpha+1})/L_\alpha$. But by Lemma 5.1.2, $(H + L_\alpha)/L_\alpha$ nci $(H + L_{\alpha+1})/L_\alpha$, hence by Lemma 5.3.5(i), $(H + L_\alpha)/L_\alpha$ asc $(H+L_{\alpha+1})L_\alpha$.

Therefore $H + L_{\alpha}$ asc $H + L_{\alpha+1}$. This is true for all $\alpha < \lambda$. So $H = H + L_0$ asc $H + L_{\lambda} = L$.

(ii) This can be proved exactly the same way as in (i) using Lemma 5.3.5 (ii). □

Next we prove the following.

<u>Theorem 5.3.6</u>

Let L be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H \leq L$. Then

(i) If H nci L, then H asc L.

(ii) If H has finite dimension, the H sci L implies H si L, while H ci L implies H asc L.

<u>Proof</u>

(i) Let S be a soluble ideal of L such that L/S is finite-dimensional. We prove that H asc L by induction on the derived length m of S. If m = 0, then L is finite-dimensional and by Theorem 5.2.1, H si L. We assume therefore that $m \geq 1$. Let A be the last non-zero term of the derived series of S. Then A is an abelian ideal of L and S/A has derived length m-1. Since (H + A)/A nci L/A it follows by induction that (H + A)/A asc L/A. Therefore H + A asc L and by Lemma 5.3.5 (i) H asc H + A. Hence H asc L.

(ii) This can be proved exactly the same way as in (i) using Lemma 5.3.5 (ii). □

Finally we prove the following which generalizes Stitzinger [28, Theorem 3].

Theorem 5.3.7

Let L be an ideally finite Lie algebra over a field of characteristic zero and let H $<$ L. Then H asc L if either of the following conditions holds:

(i) H ci L

(ii) H asc $\langle H,x\rangle$ for each $x \in L$.

The proof follows after the following lemmas.

Lemma 5.3.8

Over a field of characteristic zero, let $\mathfrak{X}$ be an S-closed Fitting class and let $L \in L\mathfrak{F}$. Then $\rho_{L\mathfrak{X}}(L)$ contains every serial $L\mathfrak{X}$-subalgebra of L.

Proof

See Amayo and Stewart [5, Theorem 13.3.7, p. 261]. □

Lemma 5.3.9

A locally nilpotent ideally finite Lie algebra is hypercentral of height $< \omega$.

Proof

See Stewart [26, Theorem 3.6, p. 40]. □

Lemma 5.3.10

Let L be an ideally finite Lie algebra over any field and let $H \leq L$. Then

(i) H ci L implies $H^{\omega} \triangleleft L$

(ii) H asc $\langle H,x\rangle$ for each $x \in L$ implies $H^{\omega} \triangleleft L$.

Proof

(i) By hypothesis we have $L = \cup I_{\alpha}$, where I_{α} is a finite-dimensional ideal of L. For each $n \in \mathbb{N}$, let $H_n = \{x \in L \mid [x,{}_nH] \subseteq H\}$. We claim that $L = \cup H_n$. Suppose not. Then $I_{\alpha} \nsubseteq H_n$ for all $n \in \mathbb{N}$. Since H idealises I_{α} and H_n, it follows that $(I_{\alpha} + H_n)/H_n$ is a non-zero finite-dimensional H-module. By assumption adh induces a nilpotent transformation on $(I_{\alpha} + H_n)/H_n$ for each $h \in H$. Hence by Lemma 5.2.6, there exists $y \in I_{\alpha} \setminus H_n$ such that $[y,H] \subseteq H_n$. Therefore $y \in H_{n+1}$ and $I_{\alpha} \cap H_n \subsetneq I_{\alpha} \cap H_{n+1}$ for all $n \in \mathbb{N}$. This is a contradiction since I_{α} is finite-dimensional. Hence $L = \cup H_n$.

Now let $x \in L$. Then $x \in H_n$ for some $n \in \mathbb{N}$ and so $[x,{}_nH] \subseteq H$. Hence for any $m > 0$, $[x,H^{\omega}] \subseteq [x,H^{n+m-1}] \subseteq H^m$. It follows that $[x,H^{\omega}] \subseteq \cap H^m = H^{\omega}$ and $H^{\omega} \triangleleft L$.

(ii) Since H asc $\langle H,x\rangle$, then $H \triangleleft^{\omega} \langle H,x\rangle$ for L is ideally finite. Let $H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_{\omega} = \langle H,x\rangle$, where $H_{\omega} = \bigcup_{i=0}^{\infty} H_i$. Let $y \in L$. Then $y \in H_n$ for some n and H si H_n. Since $H^{\omega} \triangleleft H_n$, $[y,H^{\omega}] \subseteq H^{\omega}$ and $H^{\omega} \triangleleft L$. □

<u>Lemma 5.3.11</u>

Let L be a locally finite Lie algebra over any field and let $H < L$. Then H ser L if and only if $H \cap K$ si K for every finite-dimensional subalgebra K of L.

<u>Proof</u>

See Amayo and Stewart [5, Proposition 13.2.4, p. 258]. □

<u>Proof of Theorem 5.3.7</u>

(i) Let K be any finite-dimensional subalgebra of L. Then $H \cap K$ ci K by Lemma 5.1.2 (i). By Theorem 5.2.1, we have $H \cap K$ si K, hence by Lemma 5.3.11, H ser L. By Lemma 5.3.10 (i), $H^{\omega} \triangleleft L$, so we may assume that $H^{\omega} = 0$ and H is locally nilpotent. Therefore $H \subseteq \rho(L)$ by Lemma 5.3.8. But by Lemma 5.3.9, $\rho(L)$ is hypercentral, hence H asc $\rho(L) \triangleleft L$. Therefore H asc L.

(ii) Let K be any finite-dimensional subalgebra of L, and let $x \in K$. Then H asc $\langle H,x \rangle$, so $H \cap K$ asc $\langle H,x \rangle \cap K \geq \langle H \cap K,x \rangle$. So $H \cap K$ asc $\langle H \cap K,x \rangle$. The latter has finite dimension, so $H \cap K$ si $\langle H \cap K,x \rangle$ for all $x \in K$.

By Corollary 5.2.7, we have, $H \cap K$ si K, hence by Lemma 5.3.11, H ser L. By Lemma 5.3.10 (ii), $H^{\omega} \triangleleft L$, hence we may assume that $H^{\omega} = 0$ and as in part (i) we deduce that H asc L. □

CHAPTER SIX : SUBIDEALS OF THE JOIN OF PERMUTABLE LIE ALGEBRAS

Wielandt [37] has shown that a common subnormal subgroup of two permutable subgroups of a finite group is subnormal in their product. Following [2,3], we shall obtain a similar result for Lie algebras. In particular we prove an analogue of Wielandt's theorem for finite-dimensional Lie algebras over a field of characteristic zero. Chao and Stitzinger [8] proves a similar result for finite-dimensional soluble Lie algebras in arbitrary characteristics without the permutability assumption. For insoluble Lie algebras some such hypothesis is necessary, as we show by examples. We also obtain some analogues of theorems of Wielandt [36].

Finally we extend our results to certain classes of infinite-dimensional Lie algebras. Our main results are as follows:

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$ and $A \subseteq H$, $A \subseteq K$. Then A si L if and only if A si H and A si K (Theorem 6.2.1).

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$. If H_i si $\langle H_i, H_j \rangle$ for all $i,j = 1,2,3$ and $\langle H_1, H_2 \rangle$ is permutable with H_3, then H_i si L for all i (Theorem 6.2.9). A generalization to infinite-dimensional Lie algebras leads to the following:

Let L be a Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$ and $A \subseteq H$, $A \subseteq K$. Then

(a) If L is soluble-by-finite and A si H, A si K, then A si L.

(b) If L is ideally finite and A asc H, A asc K, then A asc L (Theorems 6.3.1 and 6.3.8).

Let L be a Lie algebra over any field and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$. If $[H_1, H_2] \subseteq H_1$ and if H_i si $\langle H_i, H_j \rangle = H_i + H_j$ for all $i,j = 1,2,3$, then H_i si L for all i (Theorem 6.3.5).

Finally if L is an ideally finite Lie algebra over a field of characteristic zero and if H_1, H_2, H_3 are subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$, H_i asc $\langle H_i, H_j \rangle$ for all $i,j = 1,2,3$ and $\langle H_1, H_2 \rangle$ is permutable with H_3. then H_i asc L for all i (Theorem 6.3.10).

1. Preliminaries

Let L be a Lie algebra over any field. We recall that the *Fitting radical* $\nu(L)$ is the sum of nilpotent ideals of L (equal to the nil radical in finite dimensions). If L has finite dimension and the ground field has characteristic zero, then $\nu(L)$ contains every nilpotent subideal of L (see Amayo and Stewart [5, p. 114]). Let $H \leq L$, then we write H^L to denote the smallest ideal of L which contains H and is called the *ideal closure* of H in L. Two subalgebras H and K of a Lie algebra L are said to be *permutable* (see [5, p. 33]) if and only if $[H,K] \subseteq H + K$. If this is so, then $\langle H,K\rangle = H + K$. Let H, K be subalgebras of a Lie algebra L. We say that H and K are *cosubideal* if each of them is subideal in their join.

6.2 Finite dimensions

Theorem 6.2.1

Let L be a finite-dimensional Lie algebra over a field F of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$ and $A \subseteq H$, $A \subseteq K$. Then A si L if and only if A si H and A si K.

To prove this we need the following well-known results:

Lemma 6.2.2

Let L be a finite-dimensional soluble Lie algebra over any field and let A, H, K be subalgebras of L. If A si H and A si K, then A si $\langle H,K\rangle$.

Proof

See Chao and Stitzinger [8, Theorem 6]. □

We also need a result due to Dynkin [10], before we state it we recall the following definition:

Let L be a finite-dimensional semi-simple Lie algebra over a field of characteristic zero. A subalgebra H of L is said to be *regular* (see Dynkin [10, p.142) if there exists a basis consisting of elements of some Cartan subalgebra C of the algebra L and root vectors of the algebra L relative to C.

<u>Lemma 6.2.3</u>

Let L be a finite-dimensional semi-simple Lie algebra over algebraically closed field of characteristic zero and let Φ be a root system for L relative to a Cartan subalgebra C. Let π be a system of simple roots, $\alpha \in \pi$ and $\pi_1 = \pi \setminus \{\alpha\}$ Let δ be the smallest root of L and let
$L(\alpha) = \langle e_\delta, e_{-\delta}, e_\beta, e_{-\beta}\ (\beta \in \pi_1)\rangle$. Let
$L[\alpha] = \langle c_\alpha, e_\alpha, e_\beta, e_{-\beta}(\beta \in \pi_1)\rangle$. Then

(i) Every subalgebra $L(\alpha)$ except L is a semi-simple regular maximal subalgebra of L.

(ii) Every subalgebra $L[\alpha]$ except L is a non semi-simple regular maximal subalgebra of L.

(iii) Every regular maximal subalgebra is conjugate to one of these subalgebras.

Proof

See Dynkin [10, Theorem 5.5, p. 148]. □

Next the following.

Lemma 6.2.4

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero. Then every non-regular maximal subalgebra is semi-simple.

This result is due to Morozov and it is noted in Dynkin [10, p. 213].

Definition

The height of a root β, which is denoted by htβ, is defined to be the sum of all the coefficients in the expression of β as a linear combination of simple roots. That is if $\beta = \sum_{\alpha \in \pi} k_\alpha \alpha$ then $\mathrm{ht}\beta = \sum_{\alpha \in \pi} k_\alpha$. See Humphreys [15, p. 47].

Proof of Theorem 6.2.1

If A si L, then it is clear that A si H and A si K.

The converse comes in several stages.

(i) We may assume that the field F is algebraically closed. For if not let $\bar{F}$ be the algebraic closure of F. Then $L = H + K$ implies $\bar{L} = \bar{H} + \bar{K}$, where $\bar{L} = L \otimes_F \bar{F}$, $\bar{H} = H \otimes_F \bar{F}$ and $\bar{K} = K \otimes_F \bar{F}$, $\bar{A} = A \otimes_F \bar{F}$. Further, assuming the theorem in the algebraically closed case, we have $\bar{A}$ si $\bar{L}$. Hence $A = \bar{A} \cap L$ si L.

(ii) Assume that L is a counter-example of minimal dimension and assume that the choice of A, H, K has been made in such a way that H then has maximal dimension. Since $A^{\omega} \triangleleft H$ and $A^{\omega} \triangleleft K$, it follows that $A^{\omega} \triangleleft L$. If $A^{\omega} \neq 0$, then by minimality A/A^{ω} si L/A^{ω} and A si L, a contradiction. Therefore $A^{\omega} = 0$ and A is nilpotent. It follows that $a \in A$ implies $\langle a \rangle$ si A, by Lemma 5.2.2. If the theorem were true for the case dim $A = 1$, it would follow that $\langle a \rangle$ si L for all $a \in A$, hence $A < \nu(L)$. Therefore A would be a subideal. It follows that we may assume that dim $A = 1$, so that $A = \langle a \rangle$ for some $a \in L$.

(iii) Let $S = \sigma(L)$. If $S = L$, then A si L by Lemma 6.2.2, which is again a contradiction. Hence $S \neq L$. It follows that $S + A \neq L$. If $S \neq 0$, then $(S + A)/S$ si L/S by minimality, but dim $(S + A/S) \leq 1$ and L/S is semi-simple. Therefore $A < S$. Since ad(a) is nilpotent on H and on K. It acts nilpotently on S, so $\langle a \rangle$ si $S \triangleleft L$ which again is a contradiction. Therefore $S = 0$ and L is semi-simple.

(iv) We may assume that L is simple. For if not, then $L = L_1 \oplus L_2$ where L_1 and L_2 are ideals of L and dim $L_1 > 0$, dim $L_2 > 0$. Now $(A + L_1)/L_1$ si L/L_1 by minimality, and since dim $(A + L_1/L_1) \leq 1$ we must have $A \subseteq L_1$. Similarly $A \subseteq L_2$, so $A \subseteq L_1 \cap L_2 = 0$, contrary to hypothesis.

(v) H must be a maximal subalgebra of L. If not, let B be a maximal subalgebra of L with $H < B$. Then $B = B \cap (H + K) = H + (B \cap K)$. Since A si H and A si $B \cap K$, it follows that A si B by minimality. Now $L = B + K$ and therefore the maximal choice of the dimension of H yields $H = B$.

(vi) We now have an algebraically closed field, and a simple Lie algebra $L = H + K$, where H is a maximal subalgebra of L, $A = \langle a \rangle$ and $a \in \nu(H) \cap \nu(K)$. We claim that $a = 0$. We appeal to the classification of maximal subalgebras by Dynkin [10]. Clearly H cannot be semi-simple. Since maximal subalgebras of simple Lie algebras are regular or non-regular and since a non-regular maximal subalgebra of a simple Lie algebra is semi-simple by Lemma 6.2.4, we may assume that H is a non-semi-simple regular maximal subalgebra of L. Now by Lemma 6.2.3, H has a basis consisting of the elements $e_\alpha (\alpha \in \Phi_1^-)$, c_β $(\beta \in \pi)$ and e_γ $(\gamma \in \Phi^+)$ where Φ_1^- denotes the negative roots in $\Phi_1 = \Phi \cap \mathbb{R}\pi_1$ and Φ^+ denotes the positive roots in Φ. Also $\nu(H)$ has basis $\{e_\delta | \delta \in \Phi_2^+$ where Φ_2^+ denotes the positive roots in $\Phi_2 = \Phi \setminus \Phi_1\}$. Let L^+ be the subalgebra of H spanned by $e_\gamma (\gamma \in \Phi^+)$, and let J_n be the subspace of L spanned by H and the e_α with $\alpha \in \Phi^-$ such that $ht(-\alpha) \leq n$ where Φ^- denotes the negative roots in Φ. We see that $[J_n, L^+] < J_{n-1}$ and we have, for some n,

$H = J_0 < J_1 < \ldots < J_n = L$. These are submodules for $C + L^+$ under the adjoint representation. We have $0 \neq a \in \nu(H) \cap \nu(K)$, so we can write $a = A_\delta e_\delta + \Sigma A_\gamma e_\gamma$ with $\delta \in \Phi_2^+$ and $A_\delta \neq 0$, where the sum runs over elements $\gamma \in \Phi_2^+$ such that ht $\gamma \geq$ htδ. As $L = H + K$ we have $x = e_{-\delta} + h \in K$ for some $h \in H$, and so $\nu(K)$ contains $u = [a,x] = A_\delta e_\delta + \Sigma A_\gamma [e_{-\delta}, e_\gamma] + [a,h]$.

Here, $\Sigma A_\gamma [e_{-\delta}, e_\gamma] + [a,h] \in L^+$. Now adu acts nilpotently on the space $L/H = (H + K)/H$. But $e_{-\delta} \notin H$, so $e_{-\delta} \in J_{i+1} \setminus J_i$ for some i. Since L^+ annihilates $J_{i+1} \setminus J_i$, we have $[e_{-\delta}, u] \equiv [e_{-\delta}, A_\delta c_\delta] = 2A_\delta e_{-\delta} \bmod J_i$. So $e_{-\delta}(\text{ad } u)^r \equiv (2A_\delta)^r e_{-\delta} \bmod J_i$, a contradiction, as this can never belong to J_i. Hence $a = 0$ and A si L which is a contradiction and the argument is proved. □

Corollary 6.2.5

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$. Then A si L if and only if A si A^H and A si A^K.

Proof

If A si L, then clearly A si A^H and A si A^K.

Conversely since $A^H \triangleleft \langle A,H \rangle = H_1$, it follows that A si H_1. Similarly A si $\langle A,K \rangle = K_1$ and by Theorem 6.2.1, A si $H_1 + K_1 = L$. □

<u>Corollary 6.2.6</u>

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let A, H_i $(i = 1,\ldots,n)$ be subalgebras of L such that $L = H_1 + H_2 + \ldots + H_n$ and $A \subseteq H_i$, $\langle H_i,H_j\rangle = H_i + H_j$ for all $i,j = 1,\ldots,n$. Then A si L if and only if A si H_i for all i.

<u>Proof</u>

The proof is easily done by induction on n. □

Now we have the following which is the Lie algebra analogue of Wielandt [36, Hilfscatz 2.2].

<u>Corollary 6.2.7</u>

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let $H_i < L$ be such that H_i si $\langle H_i,H_j\rangle = H_i + H_j$ for all $i,j = 1,2,\ldots,n$. Then H_i si $L = \langle H_1,H_2,\ldots,H_n\rangle$ for each i. □

<u>Remark</u>

If L is a finite-dimensional Lie algebra over a field of characteristic zero, H, K are subalgebras of L such that $L = \langle H,K\rangle$. If A si H and A si K, then A need not be a subideal of L, as the following example shows.

Let L be the simple Lie algebra of type A_2. If $\{\alpha,\beta\}$ is a system of simple roots (in the terminology of Jacobson [16, pp. 110, 120]), and $H = \langle e_\alpha, e_\beta, e_{\alpha+\beta}\rangle$, $K = \langle e_\alpha, e_{-\beta}, e_{-\alpha-\beta}\rangle$, $A = \langle e_\alpha\rangle$ then $A \triangleleft^2 H$ and $A \triangleleft^2 K$, (see Stewart [27]). But A is not a subideal of $L = \langle H,K\rangle$.

Also if $H_1 = \langle e_\alpha\rangle$, $H_2 = \langle e_\beta\rangle$ and $H_3 = \langle e_{-\alpha-\beta}\rangle$, then $A_2 = \langle H_1,H_2,H_3\rangle$ and it can be checked that H_1,H_2,H_3 are pairwise cosubideals. But A_2 is simple, so H_i cannot be subideal for any i. This example shows that Corollary 6.2.7 is not true without the permutability assumption, however we have the following.

<u>Proposition 6.2.8</u>

Let L be a finite-dimensional soluble Lie algebra over any field. Let $A < L$ and let H_i $(i = 1,\dots,n)$ be subalgebras of L with $A < H_i$, and suppose that $L = \langle H_1,H_2,\dots,H_n\rangle$.

(i) If A si H_i for all i, then A si L.

(ii) If A si A^{H_i} for all i, then A si L

(iii) If H_i si $\langle H_i,H_j\rangle$ for all $i,j = 1,2,\dots,n$, then H_i si L for all i.

<u>Proof</u>

(i) follows from Lemma 6.2.2, and (iii) follows from (i). To prove (ii) we have A si A^{H_i} for all i, but

$A^{H_i} \lhd \langle A,H_i \rangle$, hence A si $\langle A,H_i \rangle$. Therefore by (i) A si L. □

Remark

Let L be a finite-dimensional Lie algebra over a field of characteristic $p > 0$. Clearly the *proof* of Theorem 6.2.1 fails in this case. However for *soluble* Lie algebras the result remains true by Lemma 6.2.2. The problem remains: is the result true in the insoluble case? It would seem reasonable to seek first a counter-example with L simple (even though the above reduction argument to this case cannot be justified in characteristic P). We investigated a variety of simple algebras in characteristic P. Most of the well-known simple Lie algebras do not yield such an example (at least with H maximal as one might hope possible) as may be established by a lengthy case-by-case analysis. There is, however, a counterexample using a less well-known algebra that exists only in characteristic $P = 3$, as follows.

Let F be a field of characteristic 3. Consider the Jacobson-Witt algebra W_3 over F. Then W_3 is spanned (see Frank [11]) by derivations: $A = (a_1,a_2,a_3) = a_1\Delta_1 + a_2\Delta_2 + a_3\Delta_3$, where $a_i \in F[x_1,x_2,x_3]/(x_1^3,x_2^3,x_3^3) = P$ say, and Δ_i is the derivation of P defined by $\Delta_i x_j = \delta_{ij}$. If $B = (b_1,b_2,b_3)$

$A^{H_i} \lhd \langle A,H_i\rangle$, hence A si $\langle A,H_i\rangle$. Therefore by (i) A si L. □

Remark

Let L be a finite-dimensional Lie algebra over a field of characteristic $p > 0$. Clearly the *proof* of Theorem 6.2.1 fails in this case. However for *soluble* Lie algebras the result remains true by Lemma 6.2.2. The problem remains: is the result true in the insoluble case? It would seem reasonable to seek first a counter-example with L simple (even though the above reduction argument to this case cannot be justified in characteristic P). We investigated a variety of simple algebras in characteristic P. Most of the well-known simple Lie algebras do not yield such an example (at least with H maximal as one might hope possible) as may be established by a lengthy case-by-case analysis. There is, however, a counterexample using a less well-known algebra that exists only in characteristic $P = 3$, as follows.

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multiplication in W_3 is given by $[A,B] = C = (c_1,c_2,c_3)$, where

$$c_i = \sum_j [(\Delta_j a_i)b_j - (\Delta_j b_i)a_j].$$

Let L be the subalgebra of W_3 generated by the derivations A_1, A_2, A_3, A_4, B_1, B_2, B_3, Δ_1, Δ_2, Δ_3, where $A_1 = (x_1,x_2,x_3)$, $A_2 = (0,x_2,-x_3)$, $A_3 = (x_2,x_3,0)$, $A_4 = (0,x_1,-x_2)$, $B_1 = (x_1x_2,x_1x_3, -x_2x_3)$, $B_2 = (x_1^2,x_1x_2,x_2^2)$, $B_3 = (-x_2^2,x_2x_3,x_3^2)$. Multiplication of A_i, B_i, Δ_i is given by the following:

[ ]	A_1	A_2	A_3	A_4	B_1	B_2	B_3	Δ_1	Δ_2	Δ_3
A_1	0	0	0	0	B_1	B_2	B_3	Δ_1	Δ_2	Δ_3
A_2	0	0	$-A_3$	A_4	B_1	0	B_3	0	Δ_2	$-\Delta_3$
A_3	0	A_3	0	A_1	B_3	B_1	0	0	Δ_1	Δ_2
A_4	0	$-A_4$	$-A_1$	0	$-B_2$	0	B_1	Δ_2	$-\Delta_3$	0
B_1	$-B_1$	$-B_1$	$-B_3$	B_2	0	0	0	A_3	A_1-A_2	A_4
B_2	$-B_2$	0	$-B_1$	0	0	0	0	$-(A_1+A_2)$	A_4	0
B_3	$-B_3$	$-B_3$	0	$-B_1$	0	0	0	0	A_3	A_2
Δ_1	$-\Delta_1$	0	0	$-\Delta_2$	$-A_3$	A_1+A_2	0	0	0	0
Δ_2	$-\Delta_2$	$-\Delta_2$	$-\Delta_1$	Δ_3	A_2-A_1	$-A_4$	$-A_3$	0	0	0
Δ_3	$-\Delta_3$	Δ_3	$-\Delta_2$	0	$-A_4$	0	$-A_2$	0	0	0

Clearly dim L = 10. It is shown in Frank [11] that L is simple. We use this to construct our counter-example.

Let $H = \langle A_1, A_2, A_3, A_4, B_1, B_2, B_3 \rangle$, $K = \langle A_3, \Delta_1, \Delta_2, \Delta_3 \rangle$ and $A = \langle A_3 \rangle$. Clearly $L = H + K$, and

$$A \triangleleft \langle A_3, B_3 \rangle \triangleleft \langle A_3, B_3, B_1 \rangle \triangleleft \langle A_1, A_3, B_1, B_2, B_3 \rangle \triangleleft H,$$

$$A \triangleleft \langle A_3, \Delta_1 \rangle \triangleleft \langle A_3, \Delta_1, \Delta_2 \rangle \triangleleft K,$$

but A is not a subideal of L since L is simple.

Therefore Theorem 6.2.1 does not hold in characteristic 3.

Next we prove the following which are Lie algebra analogues of Wielandt [36].

<u>Theorem 6.2.9</u>

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$ and suppose that H_i si $\langle H_i, H_j \rangle$ for all $i, j = 1,2,3$. If $\langle H_1, H_2 \rangle$ is permutable with H_3, then H_i si L for all i.

The proof will be given after the following lemmas.

<u>Lemma 6.2.10</u>

Let A, B, C be subspaces of L and let A be permutable with B and C. Then A is permutable with $\langle B, C \rangle$.

<u>Proof</u>

It is enough to show that $[A, B \circ C] \subseteq A + \langle B, C \rangle$.

Let $a \in A$, $x \in B \circ C$. Then $x = [x_1, x_2, x_3, \ldots, x_n]$, where $x_1 \in B$, $x_2 \in C$ and $x_3, \ldots, x_n \in B \cup C$. By the Jacobi identity and by induction on n,

$$[[x_1, \ldots, x_n], a] = \Sigma_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, a], x_{i+1}, \ldots, x_n] .$$

Since A permutes with B and C, it follows that for each $i = 1, \ldots, n, [x_i, a] = x_i' + a_i$ according as $x_i \in B$ or C. Therefore

$$[[x_1, \ldots, x_n], a] = \Sigma_{i=1}^{n} [x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n]$$

$$+ \Sigma_{i=1}^{n} [x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n].$$

Clearly $\Sigma_{i=1}^{n} [x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n] \in \langle B,C \rangle$. Also by the Jacobi identity and by induction on n and the permutability of A with B, C, it follows that:

$$\Sigma_{i=1}^{n} [x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n] \subseteq A + \langle B,C \rangle.$$

Hence $[A, B \circ C] \subseteq A + \langle B,C \rangle$. □

Lemma 6.2.11

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H, K be subalgebras of L such that L = H + K. If H, K are nilpotent, then L is soluble.

Proof

This follows from Kostrikin [19]. □

Lemma 6.2.12

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let $H_i < L$ be such that $L = \langle H_i | i = 1,\dots,n\rangle$. If H_i si $\langle H_i,H_j\rangle$ for all $i,j = 1,\dots,n$, then H_i si $H_i + N$ for all i, where $N = H_1^{\omega} + H_2^{\omega} + \dots + H_n^{\omega} \lhd L$.

Proof

Since H_i si $\langle H_i,H_j\rangle$, it follows that $H_i^{\omega} \lhd \langle H_i,H_j\rangle$. Therefore $H_i^{\omega} \lhd L$ and $H_1^{\omega} + \dots + H_n^{\omega} = N \lhd L$. Now the subalgebras $H_1,H_2^{\omega},\dots,H_n^{\omega}$ satisfies the hypothesis of Corollary 6.2.7. Hence H_1 si $\langle H_1,H_2^{\omega},\dots,H_n^{\omega}\rangle = H_1 + N$. The same is true for $H_2,\dots,H_n$. □

Lemma 6.2.13

Over any field of characteristic zero the class $\mathfrak{F} \cap \mathfrak{N}$ is coalescent and ascendantly coalescent.

Proof

See Amayo and Stewart [5, Theorem 2.4, p. 62]. □

Proof of Theorem 6.2.9

That H_1, H_2 are subideals of L follows from Lemma 6.2.10 and Theorem 6.2.1. To prove that H_3 si L we argue for a contradiction, assuming L to be a counter-example of minimal dimension. Consider the subalgebras H_3, H_1^ω, H_2^ω. It is clear that all pairwise permutable cosubideals and H_3 si $H_3 + N$ by Lemma 6.2.12, where $N = H_1^\omega + H_2^\omega + H_3^\omega \triangleleft L$. If $N \neq 0$, then by minimality $(H_3 + N)/N$ si L/N which implies that $H_3 + N$ si L. It follows that H_3 si L which is a contradiction. Hence $N = 0$ and all H_i are nilpotent. By Lemma 6.2.13, $\langle H_1, H_2\rangle$ is a nilpotent subideal of L. Therefore by Lemma 6.2.11, $L = \langle H_1, H_2\rangle + H_3$ is soluble and by Lemma 6.2.2 H_3 si L which is a contradiction and the theorem is proved.

<u>Remark</u>

Let L be a finite-dimensional simple Lie algebra over any field and let $H_i < L$ such that $L = \langle H_i \mid i = 1,\dots,n\rangle$. Suppose that H_i si $\langle H_i, H_j\rangle$ for all $i,j = 1,\dots,n$. Then either $H_i = L$ or H_i nilpotent.

Finally we prove the following which is the algebra analogue of Wielandt [36].

<u>Theorem 6.2.14</u>

Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H_i be subalgebras of L such that $L = \langle H_i \mid i = 1,\dots,n\rangle$. Suppose that H_i si $\langle H_i, H_j\rangle$ for all $i,j = 1,\dots,n$. If H_{i+1} is permutable with

$H_1 + H_2 + \ldots + H_i$ for all i, then H_i si L.

<u>Proof</u>

That H_1 and H_2 are subideals of L follows from Lemma 6.2.10 and Theorem 6.2.1. To prove that H_3 si L we argue for a contradiction assuming L to be a counter-example of minimal dimension, so that H_3 is not a subideal of L. Clearly $H_3^{\omega} \triangleleft L$. If $H_3^{\omega} \neq 0$, then by minimality H_3/H_3^{ω} si L/H_3^{ω} and H_3 si L which is a contradiction. Hence $H_3^{\omega} = 0$ and H_3 is nilpotent. Let $S = \sigma(L)$. If $S = L$, then H_3 si L by Lemma 6.2.2, which is again a contradiction. Hence $S \neq L$. It follows that $S + H_3 \neq L$. If $S \neq 0$, then $(H_3 + S)/S$ si L/S by minimality, but L/S is semi-simple and $(H_3 + S)/S$ is soluble, hence $H_3 + S = S$ and $H_3 < S$. Since H_3 acts nilpotently on $\langle H_i,H_3\rangle$ for all i, it follows that H_3 acts nilpotently on $H_1 + \ldots + H_n$. Therefore H_3 acts nilpotently on S, so H_3 si $S \triangleleft L$ which again is a contradiction. Therefore $S = 0$ and L is semi-simple. It follows that H_1, H_2 are ideals of L. But H_3 si $\langle H_1,H_3\rangle = H_1 + H_3$ and H_3 si $H_2 + H_3$, $H_1 + H_3$ permutes with $H_2 + H_3$ hence by Theorem 6.2.1, H_3 si $H_1 + H_2 + H_3$. Now H_3 si $\langle H_3,H_4\rangle$ and $\langle H_3,H_4\rangle$ permutes with $H_1 + H_2 + H_3$, therefore by Theorem 6.2.1 H_3 si $H_1 + H_2 + H_3 + H_4$. Continuing this process we get H_3 si L, which is a contradiction. Hence H_3 si L. By a similar argument we can show that $H_4, \ldots, H_n$ are subideals of L. □

6.3 Infinite dimensions

For infinite-dimensional Lie algebras we do not know whether a result like Theorem 6.2.1 and Theorem 6.2.9 holds in general, and we leave this as an open question. However we shall extend these results for certain classes of infinite-dimensional algebras. We start with the following:

Theorem 6.3.1

Let L be a soluble-by-finite Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$ and A si H, A si K. Then A si L.

To prove this we need:

Lemma 6.3.2

Let L be a soluble Lie algebra over any field and let $A < L$. If $[L,{}_r A] \subseteq A$ for some $r \in \mathbb{N}$, then A si L.

Proof

See Kawamoto [17, Theorem 4]. □

Lemma 6.3.3

If L is soluble-by-finite and residually nilpotent, then L is soluble.

Proof

Pick S to be a maximal soluble ideal of L and let R/S be the nilpotent residual of L/S. Let $I_\alpha \lhd L$ such that L/I_α is nilpotent and $\cap I_\alpha = 0$. Therefore $L/(I_\alpha + S)$ is nilpotent, and $R < I_\alpha + S$. Hence $R^{(m)} < I_\alpha$ and $R^{(m)} = 0$ for some $m \in \mathbb{N}$. It follows that $R = S$ and $(L/S)^\omega = 0$, but L/S is finite-dimensional, hence $(L/S)^\omega = (L/S)^n = 0$ for some $n \in \mathbb{N}$. Therefore L/S is nilpotent and L is soluble. $\square$

Proof of Theorem 6.3.1

Since A si H and A si K, it follows that $A^\omega \lhd H$ and $A^\omega \lhd K$. Therefore $A^\omega \lhd L$. Since A/A^ω is soluble-by-finite and residually nilpotent, it follows from Lemma 6.3.3 that A/A^ω is soluble. So without loss of generality we may assume that A is soluble. Let S be a soluble ideal of L such that L/S is finite-dimensional. Then $(A + S)/S$ si L/S by Theorem 6.2.1. Therefore $A + S$ si L. But $A + S$ is soluble and $[A + S,_m A] \subseteq A$ for some $m \in \mathbb{N}$, hence by Lemma 6.3.2, we have A si $A + S$. Therefore A si L. $\square$

As a consequence of Theorem 6.3.1, we have the following:

Corollary 6.3.4

Let L be a soluble-by-finite Lie algebra over a field of characteristic zero and let H_1, H_2, H_3 be subalgebras of

L such that $L = \langle H_1, H_2, H_3 \rangle$. If H_i si $\langle H_i, H_j \rangle = H_i + H_j$ for all $i,j = 1,2,3$, then H_i si L for all i. $\square$

Next we prove:

<u>Theorem 6.3</u>.5

Let L be a Lie algebra over any field and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$. Suppose that H_i si $\langle H_i, H_j \rangle = H_i + H_j$ for all $i,j = 1,2,3$ and suppose that $[H_1, H_2] \subseteq H_1$. Then H_i si L for all i.

The proof will be given after the following lemmas.

<u>Lemma 6.3</u>.6

Let L be a Lie algebra over any field and let A, H, K be subalgebras of L such that $L = H + K$ and $A \triangleleft H$, $A \triangleleft^m K$. Then $A \triangleleft^m L$.

<u>Proof</u>

Since $A^L = A^K \triangleleft L$ and $A \triangleleft^{m-1} A^K$, it follows that $A \triangleleft^m L$. $\square$

<u>Lemma 6.3</u>.7

Let L be a Lie algebra over any field and let H_1, H_2, H_3 be subalgebras of L such that $L = \langle H_1, H_2, H_3 \rangle$ and suppose that H_i si $\langle H_i, H_j \rangle = H_i + H_j$ for all $i,j = 1,2,3$. If $H_1 \triangleleft H_1 + H_2$ and $H_3 \triangleleft H_2 + H_3$, then H_i si L for all i.

Proof

That H_1, H_3 are subideals of L follows from Lemma 6.3.6.

Suppose that $H_1 \triangleleft^m H_1 + H_3$. We prove that H_2 si L by induction on m. If $m = 1$, then $H_1 \triangleleft H_1 + H_3$ and $H_1^L = H_1 \triangleleft H_1 + H_2$. But H_2 si $H_2 + H_3$, hence $H_2 + H_1$ si L and H_2 si L. Now suppose that the result is true for some $m \geq 1$. Let $I = H_1^L$, then $I = H_1^{H_3} \triangleleft H_1 + H_3$. Let $J = I \cap H_3$ and consider H_1, H_2 and J. We want to show that H_1, H_2, J are pairwise permutable cosubideals and $H_1 \triangleleft^{m-1} H_1 + J$. By assumption $H_1 \triangleleft H_1 + H_2$ and H_2 si $H_1 + H_2$. Since $I = I \cap (H_1 + H_3) = H_1 + (I \cap H_3) = H_1 + J$ and $I \triangleleft L$, it follows that $\langle H_1, J\rangle = H_1 + J$. By assumption H_1 si $H_1 + H_3$, therefore H_1 si $H_1 + J$. Also $J \triangleleft H_3$ and H_3 si $H_1 + H_3$ implies that J si $H_1 + H_3$, so that J si $H_1 + J$. Since $I \triangleleft L$ and H_2 idealises H_3, it follows that $[J, H_2] \subseteq J$. Therefore $\langle J, H_2\rangle = J + H_2$. But H_2 si $H_2 + H_3$ and $H_2 + J \subseteq H_2 + H_3$, hence H_2 si $H_2 + J$. Also $J \triangleleft H_3$ and H_3 si $H_2 + H_3$ implies that J si $H_2 + H_3$. Hence J si $H_2 + J$. Therefore H_1, H_2, J are pairwise permutable cosubideals. But $H_1 \triangleleft^{m-1} I = H_1 + J$, hence by induction H_1, H_2 and J are subideals of $H_1 + H_2 + J = H_2 + I$. Since H_2 si $H_2 + H_3$ and $I \triangleleft L$, it follows that $H_2 + I$ si $H_2 + H_3 + I = L$. Hence H_2 si L which completes the proof. $\square$

Proof of Theorem 6.3.5

Since H_3 si $H_2 + H_3$ and $H_1^L = H_1^{H_3} \lhd H_1 + H_3$, it follows that $H_1^L + H_3$ si $H_1^L + H_2 + H_3 = L$. But H_1 and H_3 are subideals of $H_1^L + H_3$, hence H_1 and H_3 are subideals of L.

Suppose that $H_3 \lhd^m H_2 + H_3$. We prove that H_2 si L by induction on m. If m = 1, then $H_3 \lhd H_2 + H_3$ which implies that H_2 idealises H_3. But H_2 idealises H_1 by assumption, hence by Lemma 6.3.7, H_2 si L. Now suppose that the result is true for some $m \geq 1$, and let $I = H_3^{H_2} \lhd H_2 + H_3$, $J = I \cap H_2$. Consider the subalgebras H_1, H_3 and J. We claim that H_1, H_3, J are pairwise permutable cosubideals and $H_3 \lhd^{m-1} J + H_3$. Since H_2 idealises H_1, it follows that $[J,H_1] = [H_2 \cap I,H_1] \subseteq H_1$ and $\langle H_1,J\rangle = H_1 + J$. Now $J \lhd H_2$ si $H_1 + H_2$ implies that J si $H_1 + H_2$. But $H_1 + J \subseteq H_1 + H_2$, hence J si $H_1 + J$. Also H_1 si $H_1 + J$. Since $I = I \cap (H_2 + H_3) = H_3+(H_2 \cap I) = H_3 + J$ and $I \lhd H_2 + H_3$, it follows that $\langle H_3,J\rangle = H_3 + J$. Also H_3 si $H_3 + J$ and J si $H_3 + J$. By assumption we have $\langle H_1,H_3\rangle = H_1 + H_3$ and H_1 si $H_1 + H_3$, H_3 si $H_1 + H_3$. Therefore H_1, J, H_3 are pairwise permutable cosubideals. But $H_3 \lhd^{m-1} I = H_3 + J$, hence by induction it follows that J si $H_1 + J + H_3 = H_1 + I$. In particular H_1, J, H_3 being subideals of $H_1 + I$ implies $I = J + H_3$ si $H_1 + I$. Now consider the subalgebras H_1, H_2 and I. We know that H_2 idealises H_1 and H_1, H_2 are subideals of $H_1 + H_2$ by assumption.

H_2 idealises I by the definition of I, so $\langle H_2, I\rangle = H_2 + I$. Also H_2 si $H_2 + I$, since H_2 si $H_2 + H_3$ and $H_2 \subseteq H_2 + I$. Further, $I \triangleleft H_2 + H_3$ and $I \subseteq H_2 + I$ implies $I \triangleleft H_2 + I$. Thus H_1 si $H_1 + I$ and I si $H_1 + I$. Therefore by Lemma 6.3.7, H_2 si $H_1 + H_2 + I = L$ which completes the proof. □

In the rest of this section we shall investigate ascendant subalgebras of the join of permutable subalgebras of ideally finite Lie algebras.

We start with the following which generalizes Stitzinger [28, Lemma 2].

<u>Theorem 6.3.8</u>

Let L be an ideally finite Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that $L = H + K$, and $A \subseteq H$, $A \subseteq K$. If A asc H and A asc K, then A asc L.

<u>Proof</u>

By assumption we have $L = \cup I_\alpha$ where I_α is a finite-dimensional ideal of L for each α. Let $F_\alpha = C_L(I_\alpha)$. Then since $A \triangleleft A + \zeta_1(L)$, we may assume that $\zeta_1(L) = 0$ and $\cap F_\alpha = 0$. Since L/F_α is finite-dimensional, it follows that $(A + F_\alpha)/F_\alpha$ si L/F_α by Theorem 6.2.1. Hence $A + F_\alpha$ si L for each α. Now given I_α we can find F_β such that $I_\alpha \cap F_\beta = 0$ for if not then $I_\alpha \cap C_L(I_\beta) \neq 0$, so that $I_\alpha \cap \zeta_1(L) \neq 0$ which is a contradiction. Since $A \cap I_\alpha \subseteq A + F_\beta$ and

$[A \cap I_\alpha, A + F_\beta] \subseteq A \cap I_\alpha$, it follows that $A \cap I_\alpha \triangleleft A + F_\beta$. But $A + F_\beta$ si L, hence $A \cap I_\alpha$ si L and $A \cap I_\alpha$ si I_α for each α. Let X be any finite-dimensional subalgebra of L. Then since L is ideally finite, it follows that X is contained in a finite-dimensional ideal of L. Therefore $A \cap X$ si X. Hence by Lemma 5.3.11, A ser L. But $A^\omega \triangleleft L$, so we may assume that $A^\omega = 0$. Now $A \subseteq \rho(L)$ by Lemma 5.3.8. By Lemma 5.3.9, $\rho(L)$ is hypercentral. Therefore A asc L. Indeed $A \triangleleft^\omega L$. □

An immediate consequence of Theorem 6.3.8, we have the following:

Corollary 6.3.9

Let L be an ideally finite Lie algebra over a field of characteristic zero and let H_i $(i = 1,2,\ldots,n)$ be subalgebras of L such that $L = \langle H_i \; i = 1,\ldots,n\rangle$. Suppose H_i asc $\langle H_i,H_j\rangle = H_i + H_j$ for all $i,j = 1,\ldots,n$. Then H_i asc L for all i. □

Next we prove the following:

Theorem 6.3.10

Let L be an ideally finite Lie algebra over a field of characteristic zero and let H_1,H_2,H_3 be subalgebras of L such that $L = \langle H_1,H_2,H_3\rangle$ and H_i asc $\langle H_i,H_j\rangle$ for all $i,j = 1,2,3$. If $\langle H_1,H_2\rangle$ is permutable with H_3, then

H_i asc L for all i.

Proof

That H_1, H_2 are ascendant in L follows from Lemma 6.2.10 and Theorem 6.3.8. By assumption we have $L = \cup I_\alpha$, where I_α is a finite-dimensional ideal of L for each α. Let $F_\alpha = C_L(I_\alpha)$. Then since $H_3 \triangleleft H_3 + \zeta_1(L)$ we may assume that $\zeta_1(L) = 0$ and $\cap F_\alpha = 0$. Since L/F_α is finite-dimensional, it follows that from Theorem 6.2.9 that $(H_3 + F_\alpha)/F_\alpha$ si L/F_α. Hence $H_3 + F_\alpha$ si L. Now argue as in Theorem 6.3.8, we get H_3 asc L. □

Finally we prove the following:

Theorem 6.3.11

Let L be an ideally finite Lie algebra over a field of characteristic zero and let $H_i \leq L$ such that $L = \langle H_i \mid i = 1,\ldots,n\rangle$ and H_i asc $\langle H_i, H_j\rangle$ for all $i,j = 1,\ldots,n$. If H_{i+1} is permutable with $H_1 + H_2 + \ldots + H_i$ for all i, then H_i asc L.

Proof

By Lemma 6.2.10 and Theorem 6.3.8 we have H_1, H_2 ascendant in L. To show that H_i asc L, $i \geq 3$ argue as in Theorem 6.3.8 and apply Theorem 6.2.14. □

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